Université de Montréal

NCAR

Solving the Monge-Ampère differential equation in the context of semi-Lagrangian advection schemes

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EULAG workshop, Septemper 6-10 2008, Bad Tölz, Germany

Outline

- Introduction and motivation
- Advection in EULAG
- A Semi-Lagrangian scheme
- MA and mass continuity equations
- Results
- Conclusions

- **Task**

  Solve a Monge-Ampere equation in the context of semi-Lagrangian advection for a variety of flows

- **Motivation**

  To improve consistency between semi-Lagrangian integration of fluid PDEs and the mass continuity equation

  (e.g: intensive variables such as pot. temperature, momentum per unit of mass, passive tracer, etc...)

- **Expectations**

  Observe improved conservation of advected scalars

  (beginning with passive advection for incompressible fluids in a cartesian geometry using periodic b.c.)

1.Advection in Eulag

- Approximate integrals for fluid PDEs

$$\Psi^{(n+1)} = A(\Psi^{n} + 0.5\,\Delta t\, R^{(n)}) + 0.5\,\Delta t\, R^{(n+1)}$$

- A= Eulerian/ semi-Lagrangian operator
- Passive advection: R=0

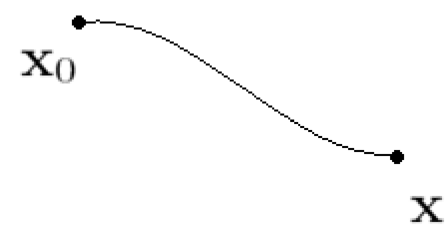
$$\Psi^{(n+1)} = A(\Psi^{n})$$

3.The semi-Lagrangian approach

- Lagrangian evolution equation

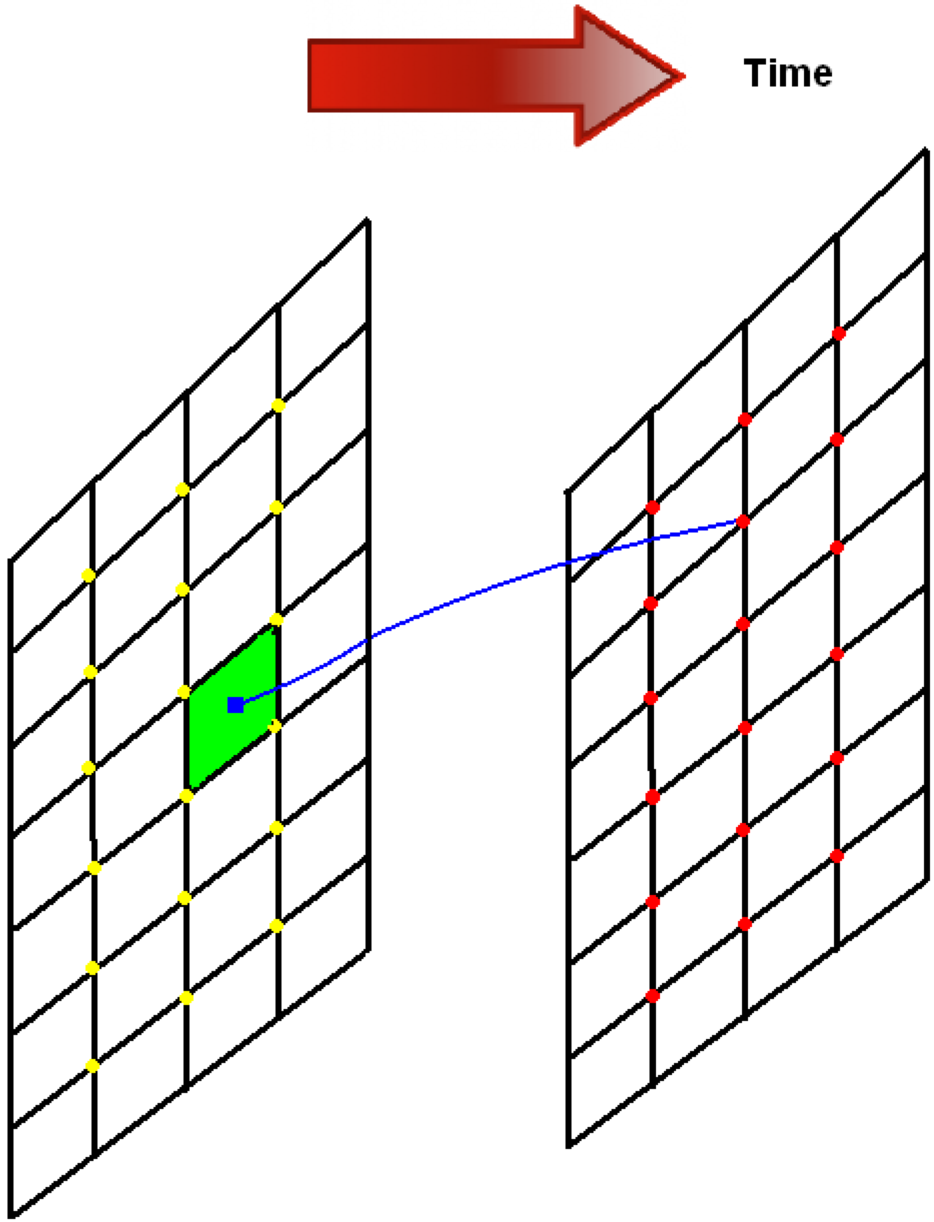
$$\psi(\mathbf{x},t) = \psi(\mathbf{x}_0,t_0) + \int_T R\,\mathrm{d}t$$

- Involves
  - Interpolation
  - Integration along the trajectory
- Passive advection: interpolation only

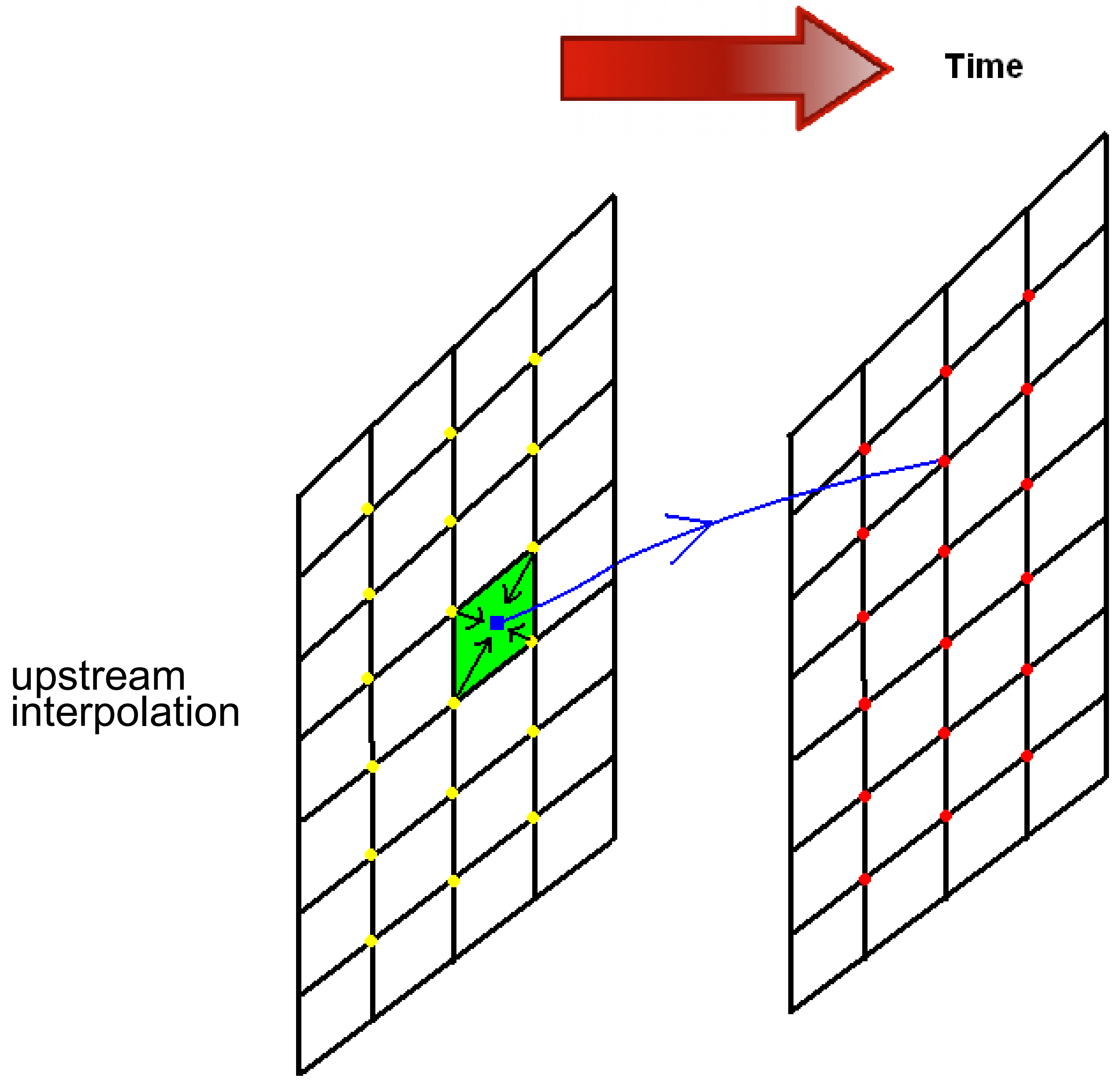


$$\Psi^{(n+1)} = A(\Psi^n)$$

- Requires knowledge of departure points



Finfing
departure
points


Time
upstream
interpolation

- Allows for the use of a constant mesh throughout advection (Eulerian aspect) by relabelling Lagrangian particles
- Time-step is not constrained by CFL (Lagrangian aspect)
- **Defect**: Typically, SL schemes based on upstream interpolation are not conservative since mass continuity equation is not accounted for (e.g. as in conservative remappings)
- ***Remedy***: Solve a Monge-Ampère equation to improve consitency between mass continuity equation and departure points

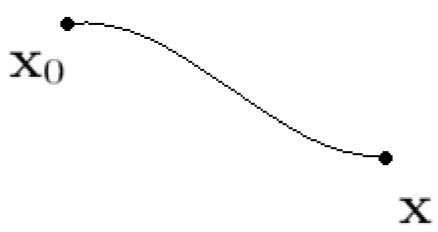
4.The mass continuity and Monge-Ampère equations

- Consider the flow map and its Jacobian

$$\mathbf{x} - \mathbf{x}_0 = \int_{t_0}^{t} \mathbf{v}(\mathbf{x}, t')\, \mathrm{d}t' \qquad \mathrm{J} := \det\left\{ \frac{\partial \mathbf{x}_0}{\partial \mathbf{x}} \right\}$$

- Note that J=1 at initial time
- Differentiate J with respect to time

$$\frac{\partial \ln \mathrm{J}}{\partial t} = \nabla \cdot \mathbf{v}$$



- For an incompressible fluid,

$$\frac{\partial \ln \mathrm{J}}{\partial t} = 0$$

- J=1 at all time and everywhere inside the fluid
- geometrical meaning : volume of fluid parcels
- However, the flow map

$$(\mathbf{x}, t) \longrightarrow (\mathbf{x}_0, t_0)$$

is not accurate. Indeed,

$$\mathbf{x}_0 \approx \mathbf{x} - \int_{t_0}^{t} \mathbf{v}(\mathbf{x}, t')\, \mathrm{d}t'$$

- Consequence & link with mass continuity

$$\mathrm{J} := \det\left\{ \frac{\partial \mathbf{x}_0}{\partial \mathbf{x}} \right\} \neq 1$$

- So that J comes *closer* to unity, find the solution to

$$J = det\left\{\frac{\partial(\mathbf{x_0} + \nabla\phi)}{\partial \mathbf{x_i}}\right\} = 1$$

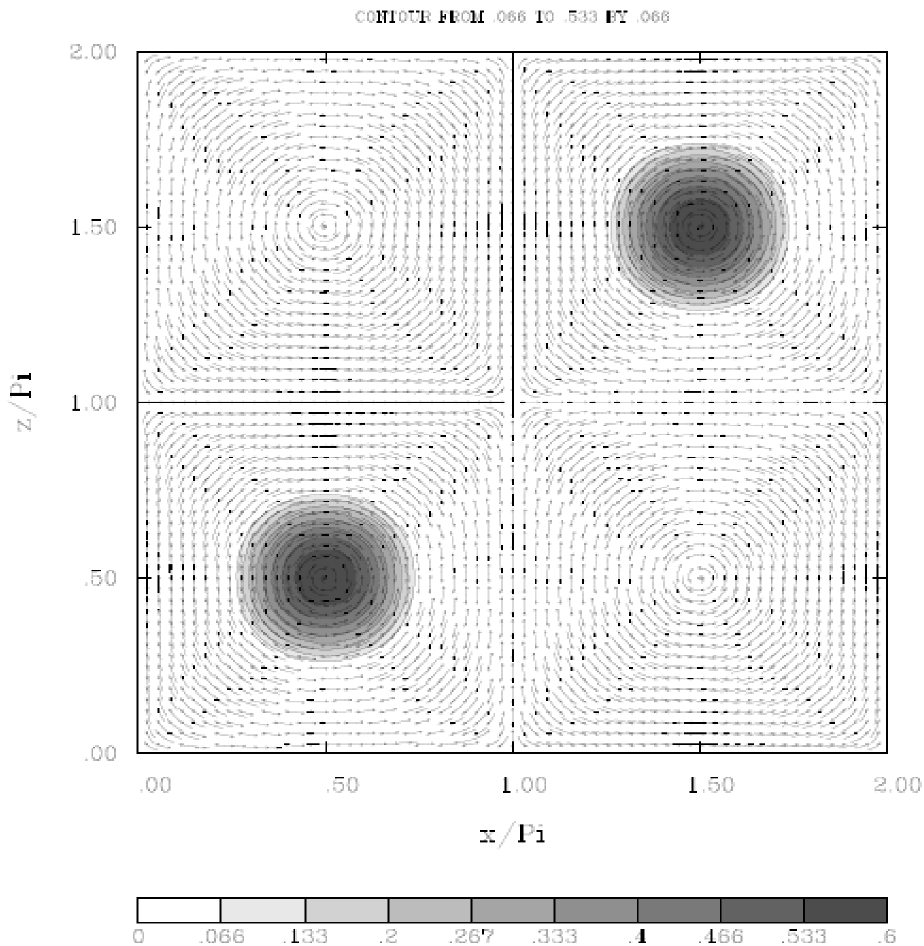
- e.g in 2D, we have to solve the Monge-Ampère equation:

$$a\frac{\partial^2\phi}{\partial x^2} + 2b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + e\left(\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right) + d = 0$$

- Nonlinear, elliptic , 2^{nd} order PDE
- This form of MA :Rare in literature -> Hessian

5.Results

- *2D passive advection*

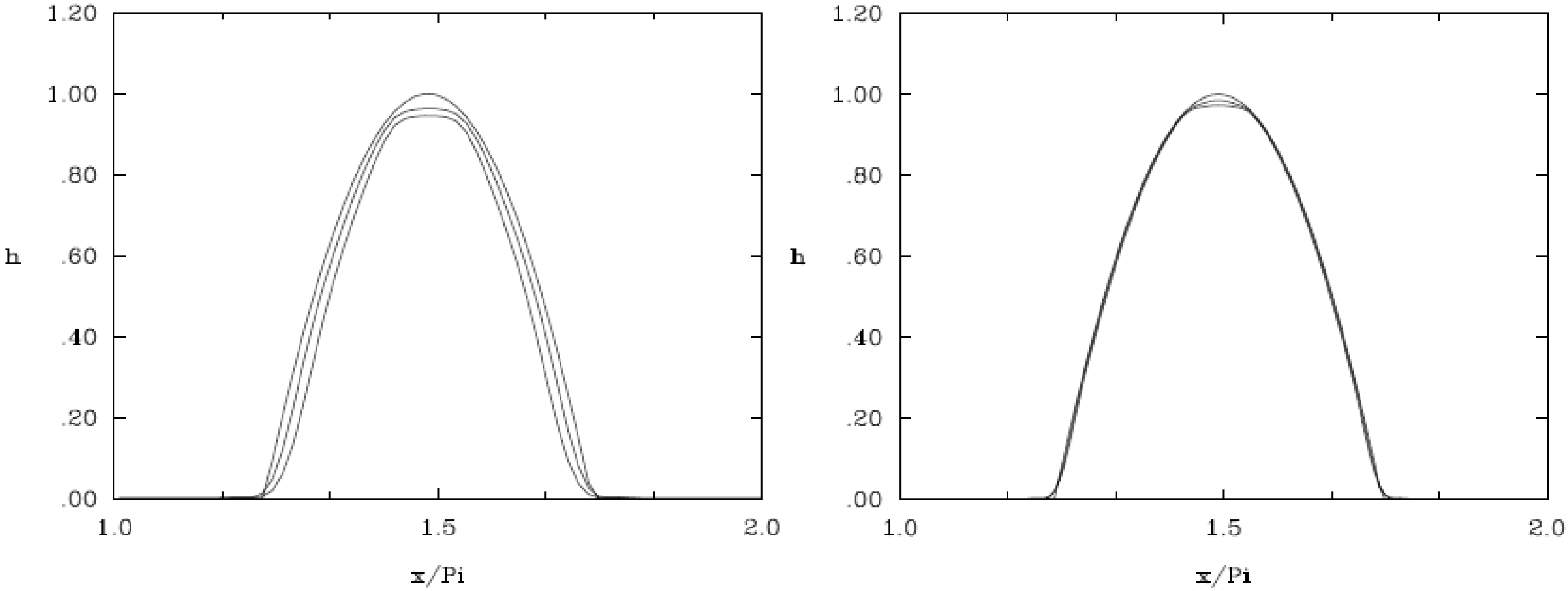


$$\Psi = \sin(x)\sin(y)$$

$$u = \frac{\partial \Psi}{\partial y} = \sin(x)\cos(y)$$

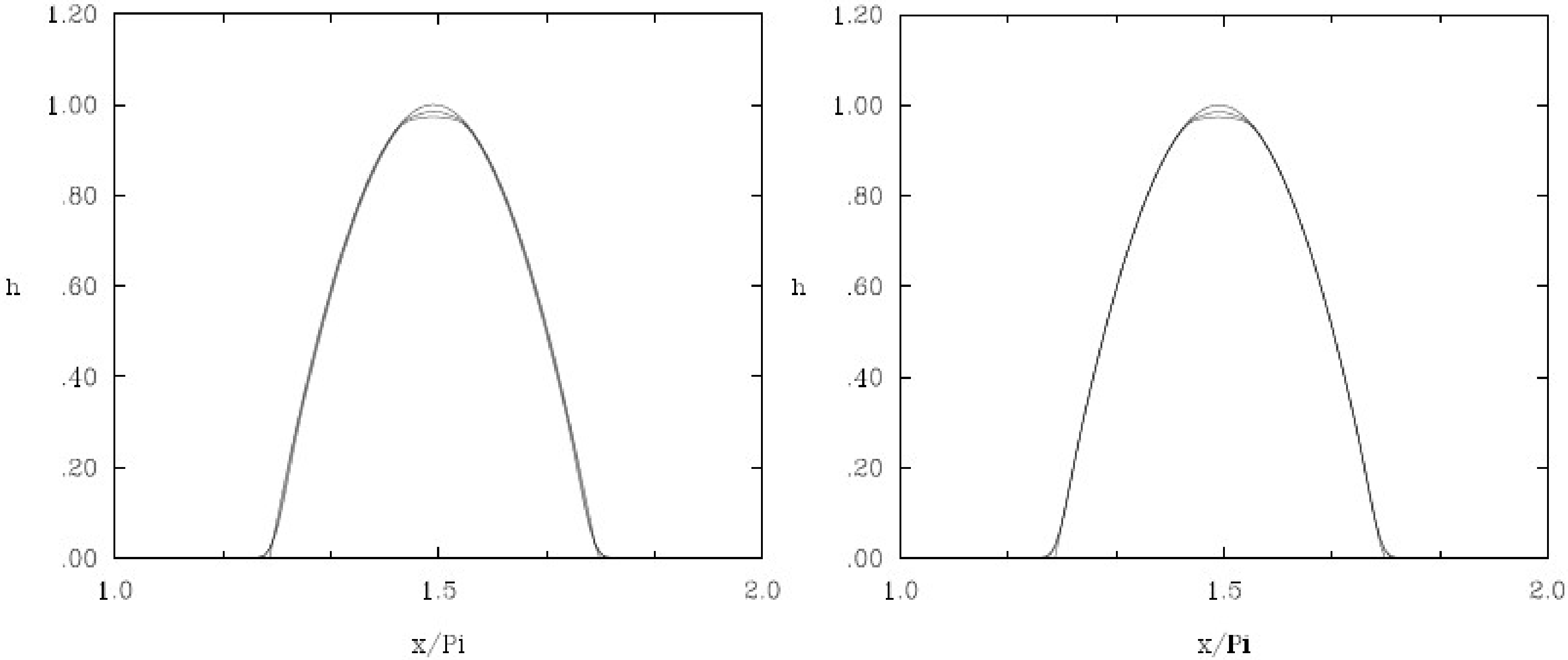
$$v = -\frac{\partial \Psi}{\partial x} = \cos(x)\sin(y)$$

- *2D passive advection*

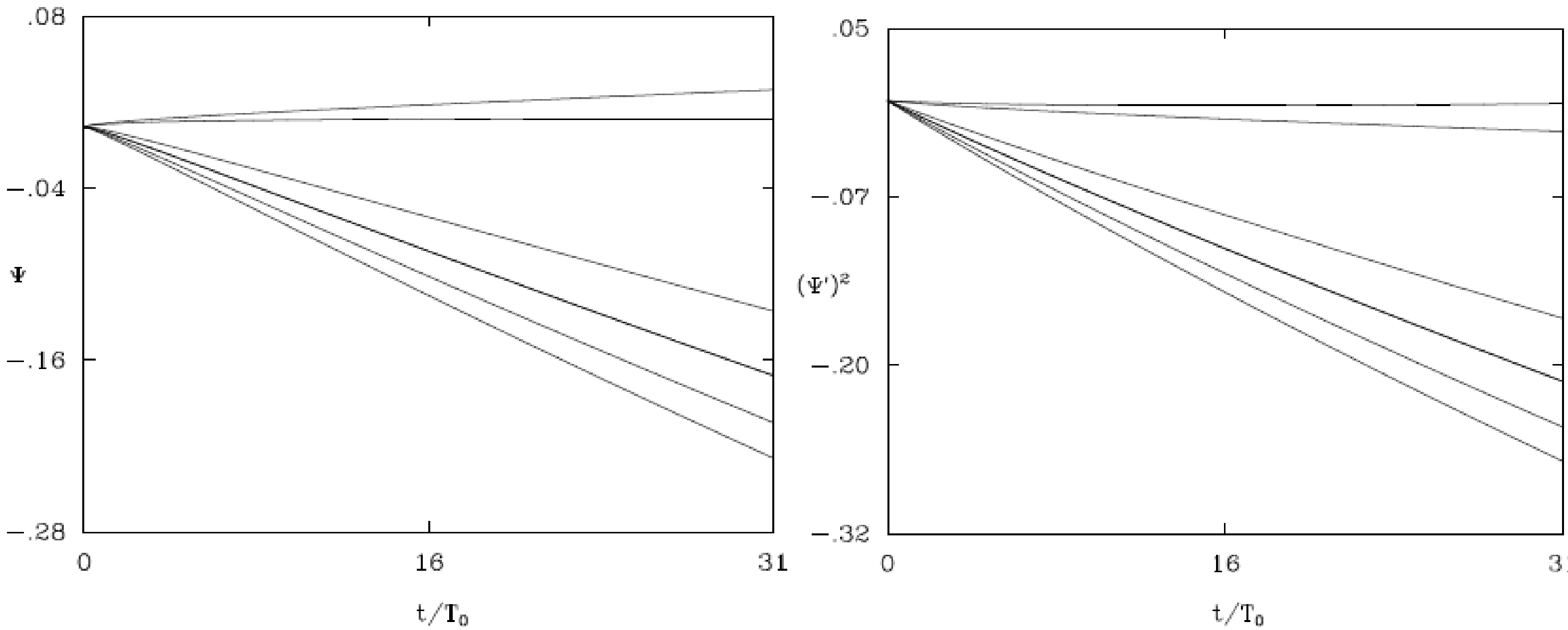


- Enhanced tracer conservation
- Accuracy of J=1 to round-off error
- Courant number=4

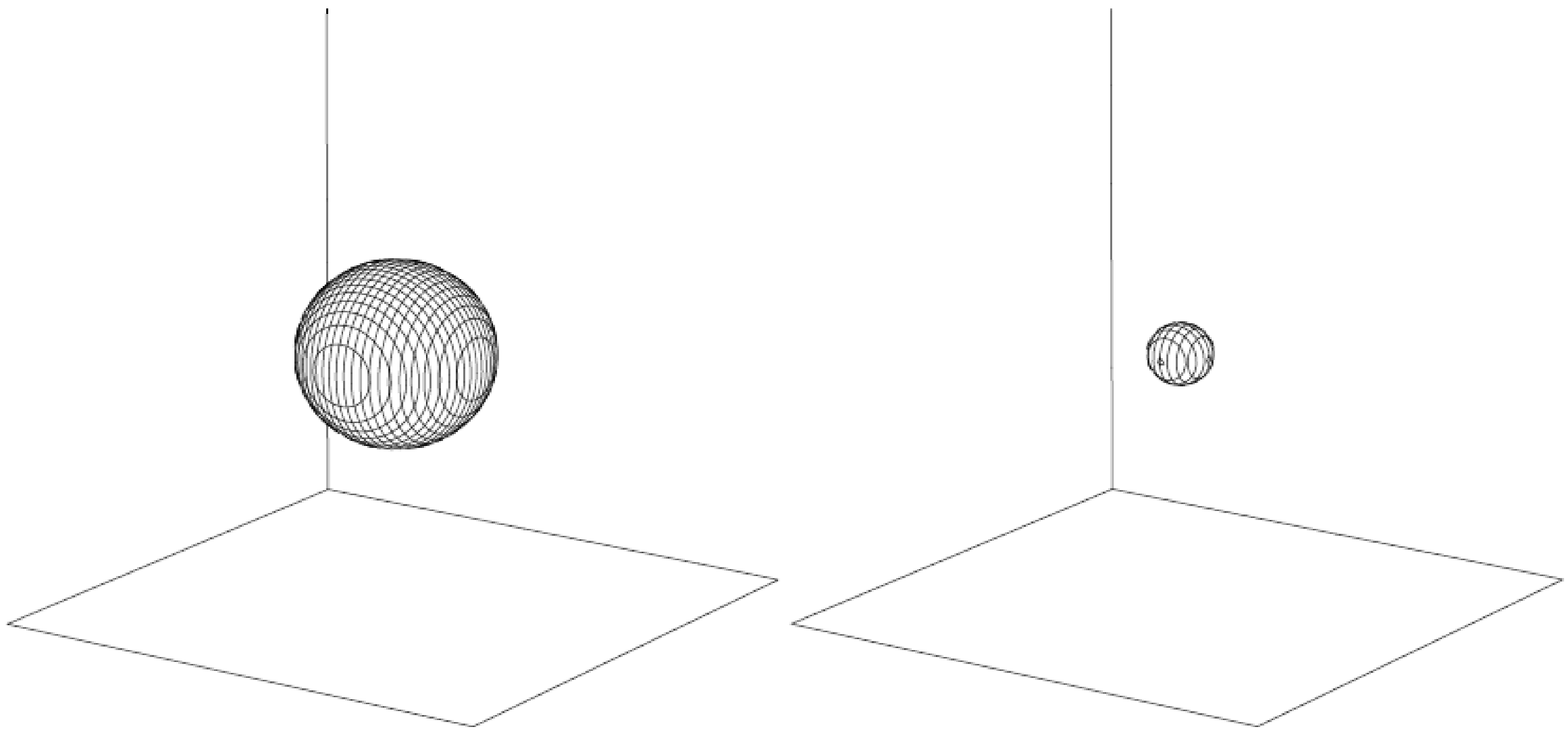
- *2D passive advection*



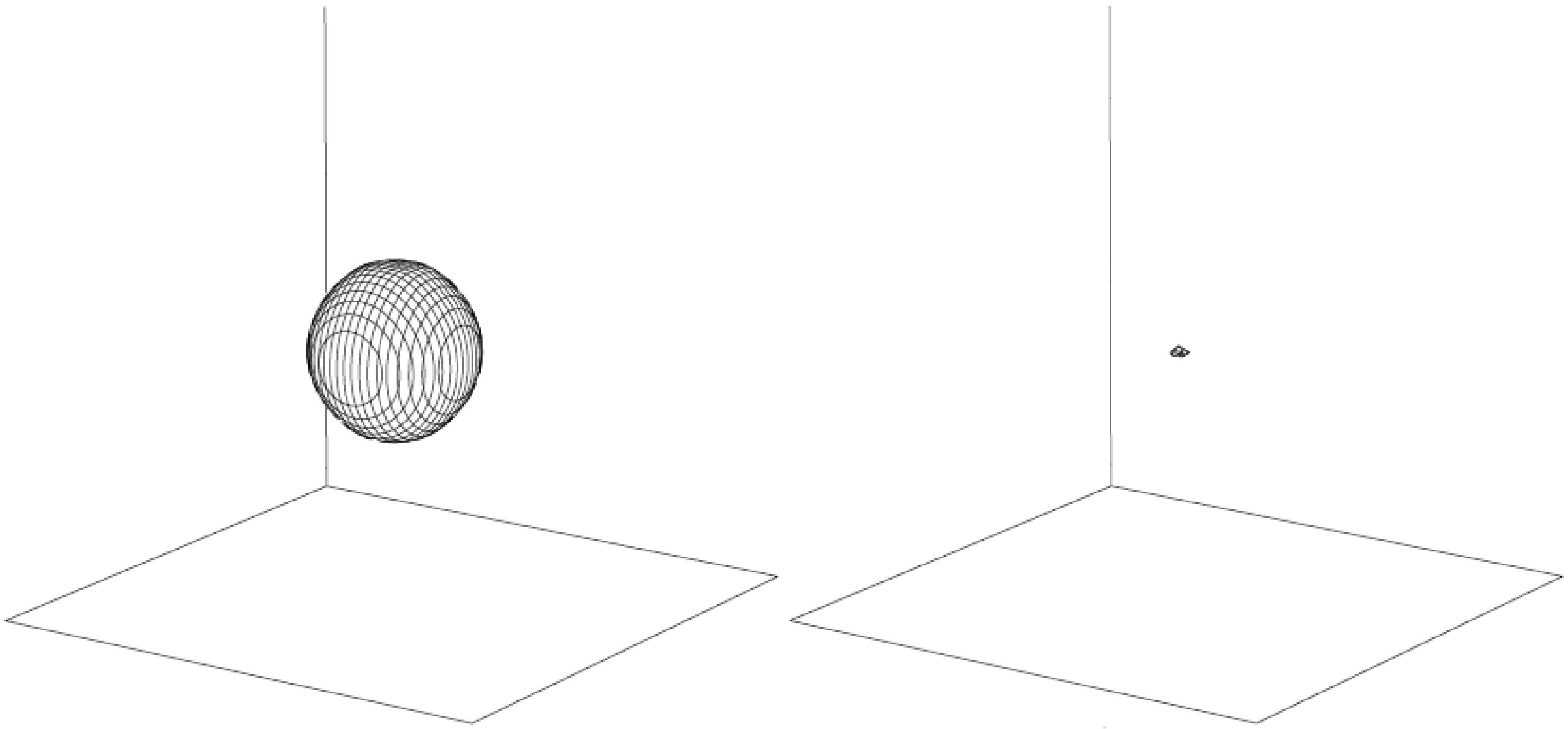
• Convergence to solution



- *3D passive advection*



- *Semi-Lagrangian*



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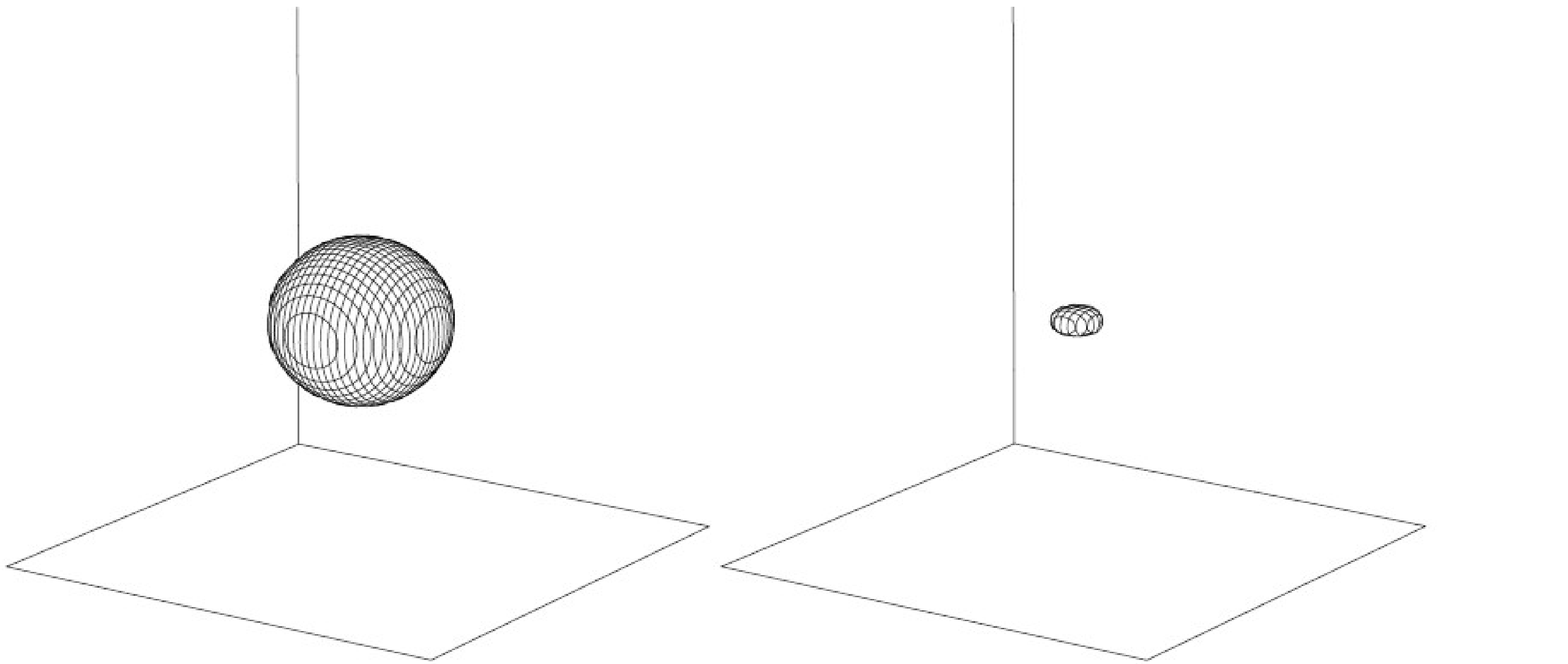
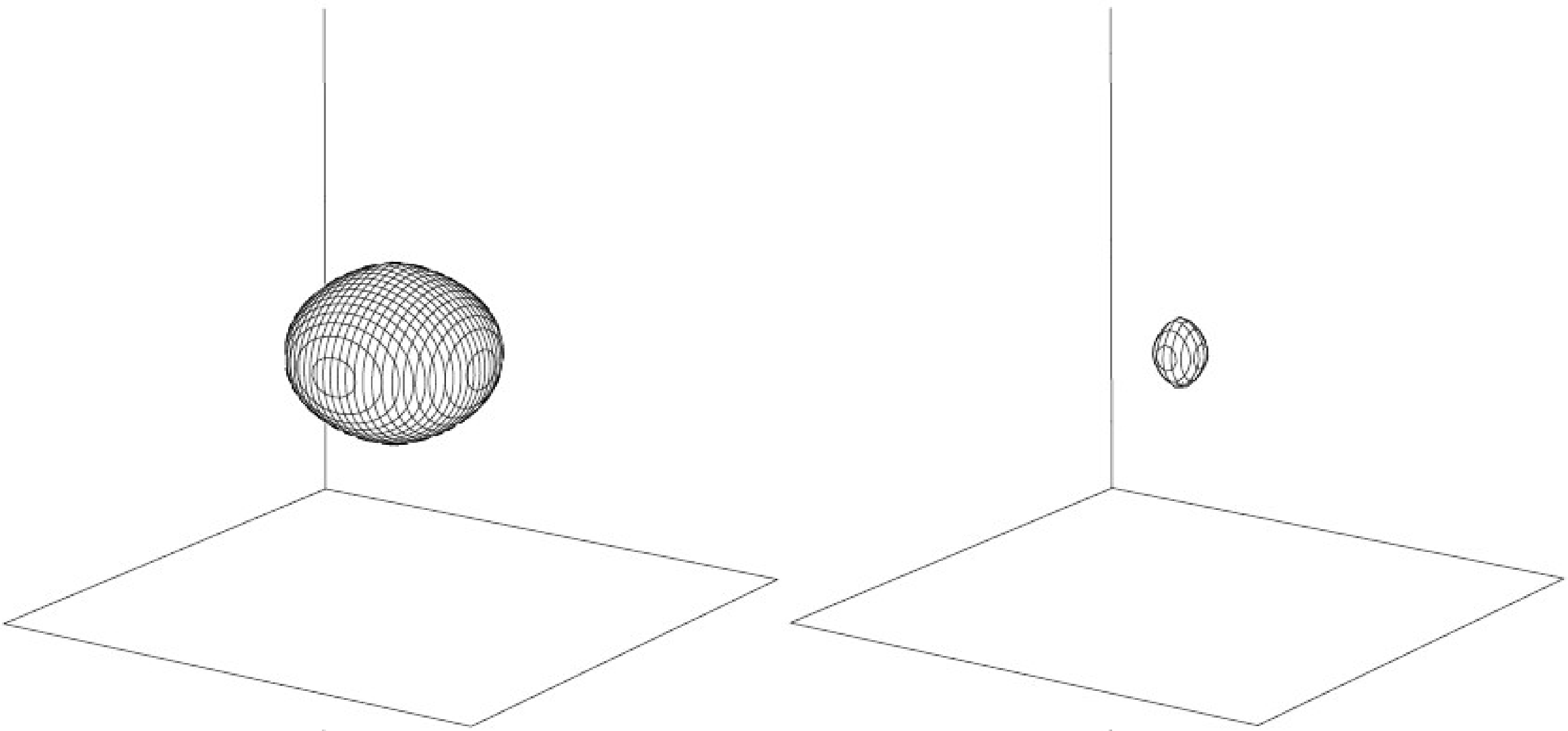
- *Semi-Lagrangian + MA*

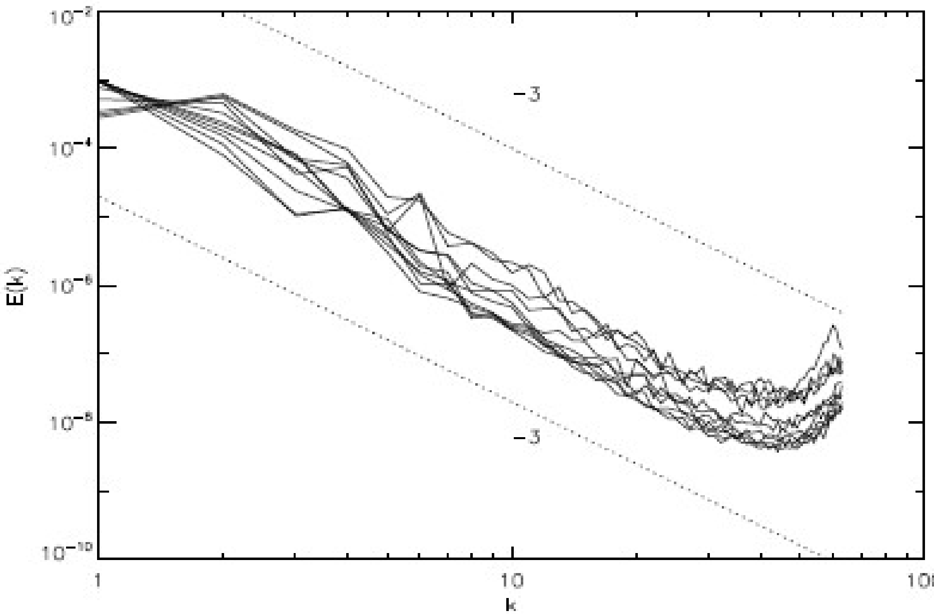


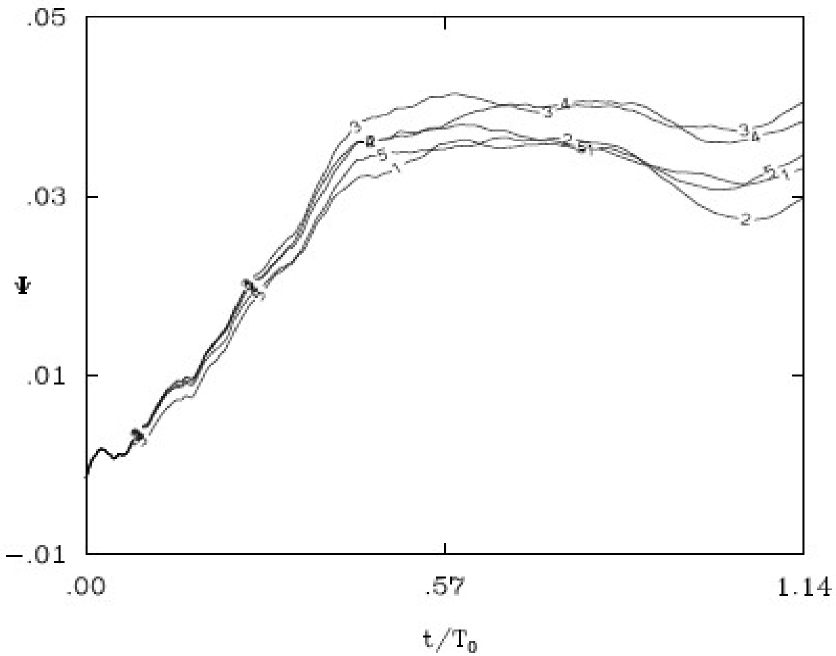
- *Eulerian*



- *2D turbulence (ILES)*

- Increasing Jacobian accuracy doesn't seem to have any effect





- Possible explanation:sharp gradients in small scales implies poor interpolation
- In turn, interpolation overwhelms MA improvement

6.Summary

- Faster MA solver (to preserve efficiency of SL scheme)
- Variational formulation
- Failure in turbulent flows
- conservative mappings
- Expanding possibilities: full compressible version of MA solver in curvilinear coordinates (e.g., climate modelling, Solar MHD)
- Grid adaptivity

Thanks!