On forward in time differencing: an unstructured mesh framework

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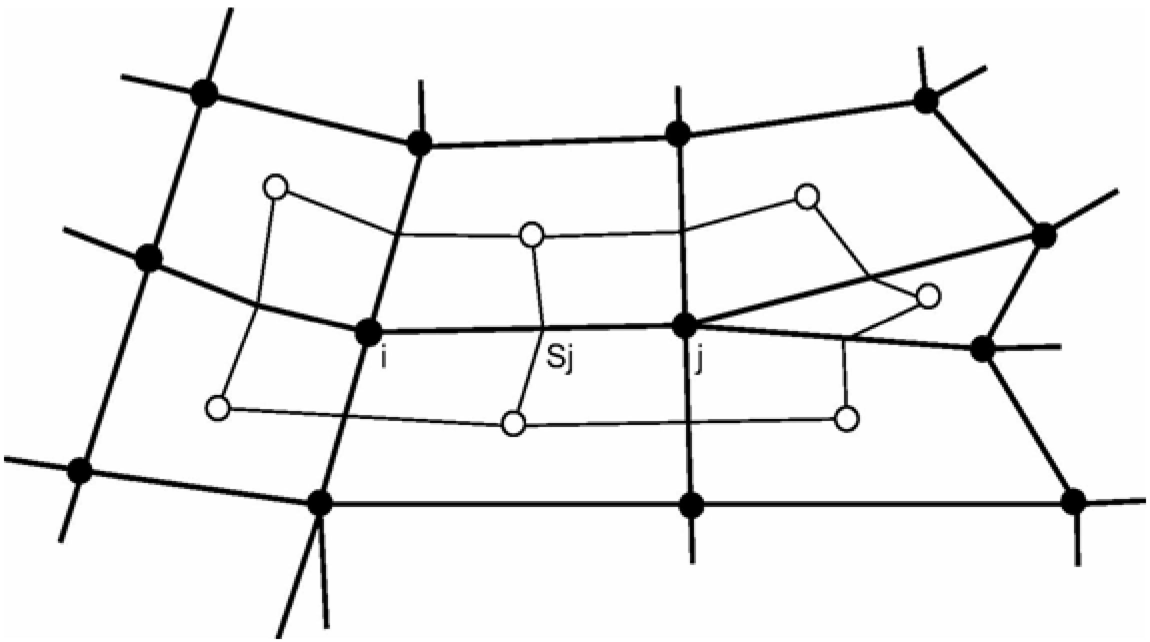
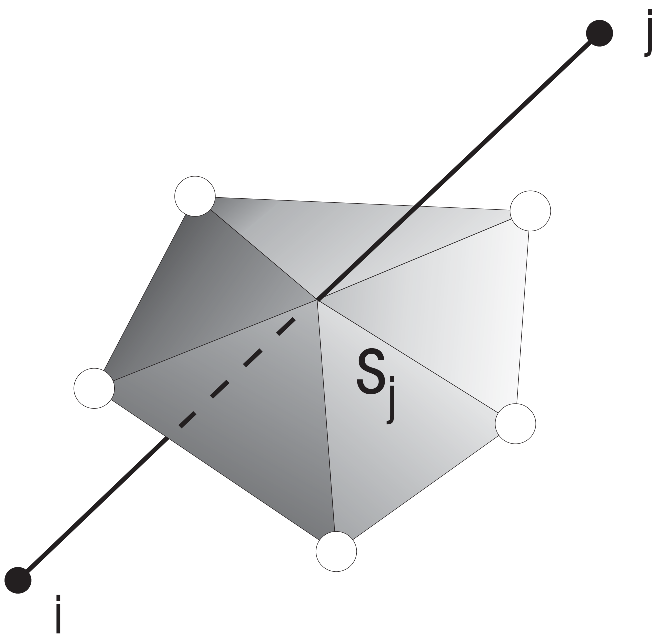
Edge based data
Finite Volume
(Smolarkiewicz, Szmelter JCP 2005)

2D



3D



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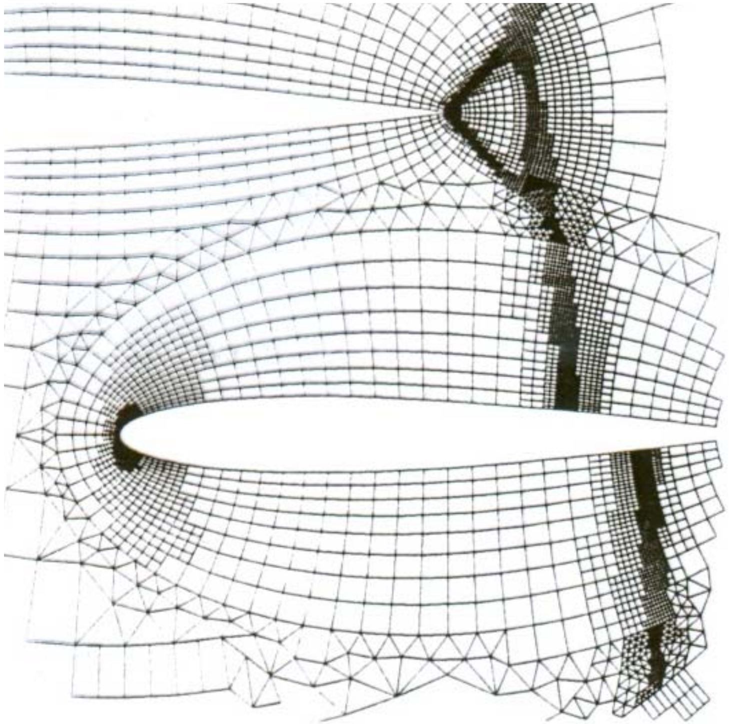
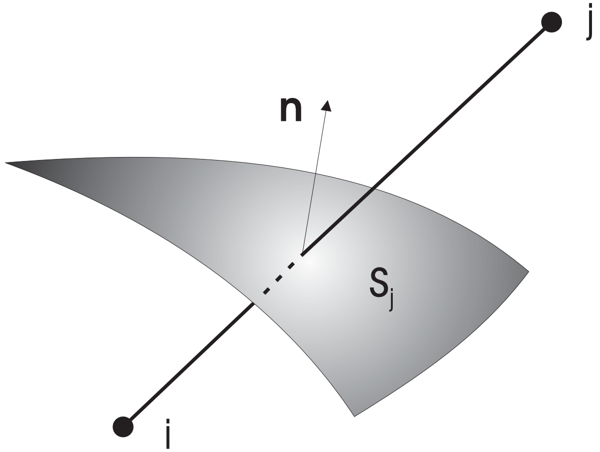
Edge based data structure

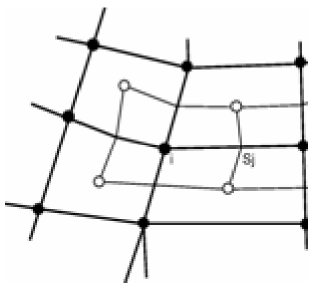
- ☺ *Flexible mesh adaptivity and hybrid meshes*
- ☺ *Low storage*
- ☺ *Easy generalisation to 3D, vectorisation, parallelisation, mesh agglomeration*
- ☺ *Less expensive than element based data structure*
- ☹ *More expensive operations than I,J,K*



Notion of the edge-based MPDATA: ADVECTION



$$\frac{\partial \Psi}{\partial t} = -\nabla \cdot (\mathbf{v}\Psi)$$



$$\Psi_i^{n+1} = \Psi_i^n - \frac{\delta t}{\mathcal{V}_i} \sum_{j=1}^{l(i)} F_j^{\perp} S_j$$

$$F_j^{\perp} = [v_j^{\perp}]^{+} \Psi_i^n + [v_j^{\perp}]^{-} \Psi_j^n$$

$$[v]^{+} := 0.5(v + |v|) \quad , \quad [v]^{-} := 0.5(v - |v|)$$

FIRST ORDER UPWIND (DONOR CELL)

$$F_j^{\perp} = v_j^{\perp} \Psi|_{s_j}^{n+1/2} + Error$$

$$\tilde{v} := -\frac{1}{\Psi} Error$$

compensating velocity

$$Error = -0.5|v_j^{\perp}| \frac{\partial \Psi}{\partial r}\Big|_{s_j}^{*} (r_j - r_i) + 0.5 v_j^{\perp} \frac{\partial \Psi}{\partial r}\Big|_{s_j}^{*} (r_i - 2r_{s_j} + r_j)$$

$$+0.5\delta t v_j^{\perp} (\mathbf{v}\nabla\Psi)|_{s_j}^{*} + 0.5\delta t v_j^{\perp} (\Psi \nabla \cdot \mathbf{v})|_{s_j}^{*} + \mathcal{O}(\delta r^2, \delta t^2, \delta t \delta r)$$

MPDATA BASIC SCHEME

ITERATED UPWIND

- Calculate normal velocity $v_j^{\perp}|$
- Calculate fluxes

$$F_j^{\perp} = [v_j^{\perp}]^{+}\Psi_i^n + [v_j^{\perp}]^{-}\Psi_j^n$$

- Update field

$$\Psi_i^* = \Psi_i^n - \frac{\delta t}{\mathcal{V}_i}\sum_{j=1}^{l(i)} F_j^{\perp}$$

FIRST ORDER UPWIND

- Calculate antidiffusive pseudo velocity

$$\hat{v}_j^{\perp} = |v_j^{\perp}|\frac{|\Psi_j^*| - |\Psi_i^*|}{|\Psi_j^*| + |\Psi_i^*| + \varepsilon} - \frac{\delta t}{2}v_j^{\perp}\left(\mathbf{v}\cdot\frac{\nabla|\Psi^*|}{\overline{|\Psi^*|}} + \nabla\cdot\mathbf{v}\right)_{S_j}$$

- Calculate fluxes

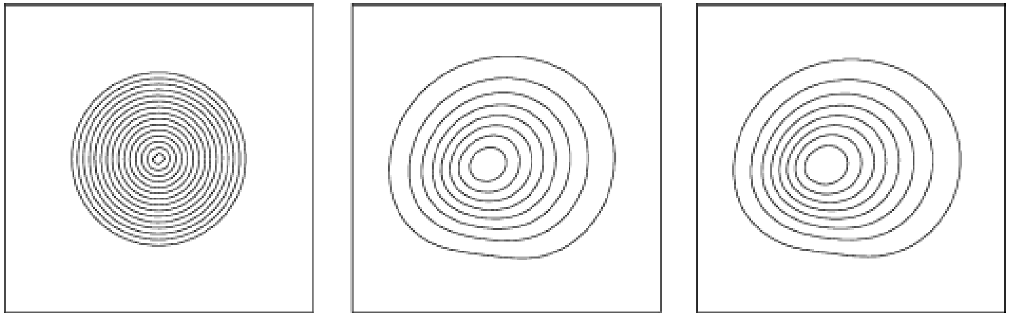
$$F_j^* = [\hat{v}_j^{\perp}]^{+}\Psi_i^* + [\hat{v}_j^{\perp}]^{-}\Psi_j^*$$

- Update field

$$\widetilde{\Psi}_i^{n+1} = \widetilde{\Psi}_i^* - \frac{\delta t}{\mathcal{V}_i}\sum_{j=1}^{l(i)} F_j^*$$

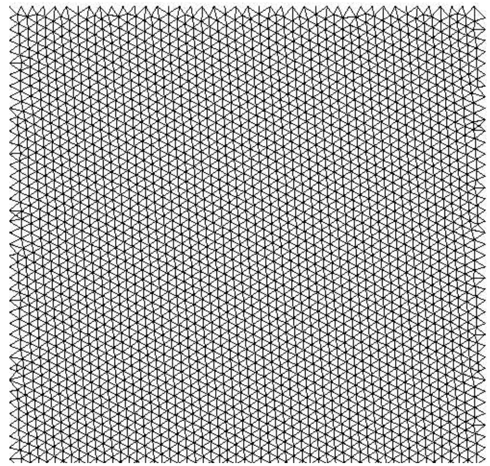
FIRST ORDER UPWIND

Structured v Unstructured



Scheme	Max	Min	L_∞	L_2
MPDATA FD	2.18	0.	2.00	0.47×10^{-3}
MPDATA FV squares	2.18	0.	2.00	0.47×10^{-3}
MPDATA FV triangles, $\delta t = 0.1$	2.19	0.	1.92	0.47×10^{-3}
MPDATA FV triangles, $\delta t = 0.061$	2.08	0.	2.09	0.52×10^{-3}
Upwind FD	0.27	0.	3.76	1.21×10^{-3}
Upwind FV squares	0.28	0.	3.76	1.04×10^{-3}
Upwind FV triangles	0.25	0.	3.68	1.06×10^{-3}
Leapfrog FD	3.16	−0.62	1.68	0.62×10^{-3}
Leapfrog FV squares	3.11	−0.67	1.71	0.64×10^{-3}
Leapfrog FV triangles	3.11	−0.69	1.74	0.65×10^{-3}

CONVERGENCE OF FINITE-VOLUME MPDATA ON UNSTRUCTURED MESH



background spacing	L_∞	L_2	L_1
2.00	$2.95 \cdot 10^{-2}$	$8.73 \cdot 10^{-3}$	$1.83 \cdot 10^{-3}$
1.00	$7.69 \cdot 10^{-3}$	$2.22 \cdot 10^{-3}$	$4.73 \cdot 10^{-4}$
0.50	$1.93 \cdot 10^{-3}$	$5.57 \cdot 10^{-4}$	$1.19 \cdot 10^{-4}$
0.25	$4.86 \cdot 10^{-4}$	$1.43 \cdot 10^{-4}$	$3.07 \cdot 10^{-5}$

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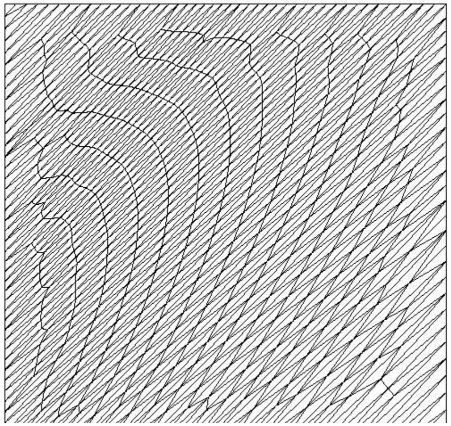
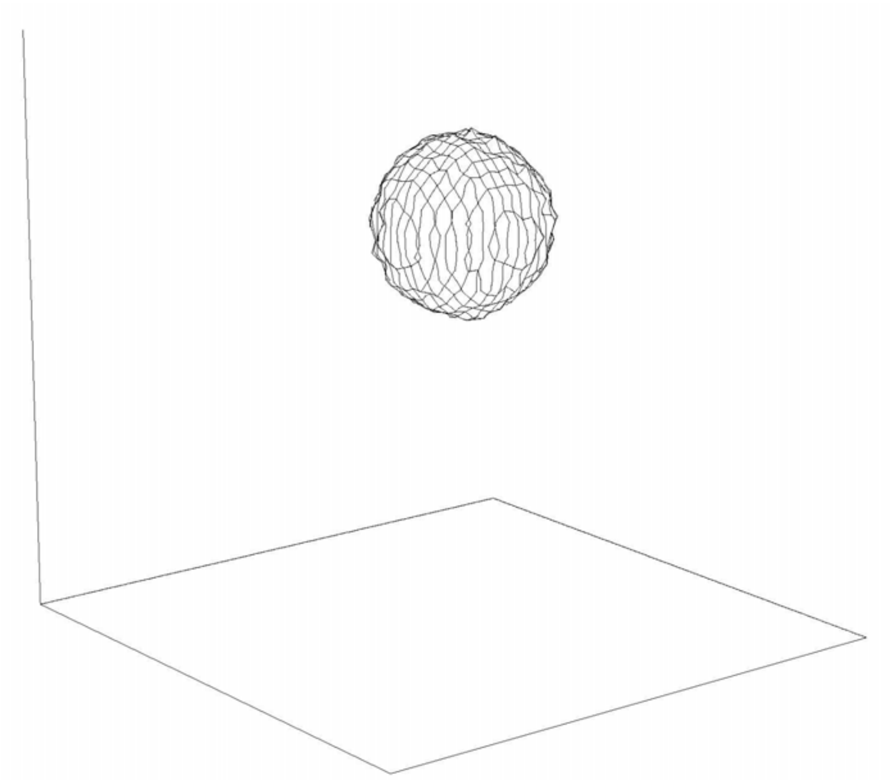
CONVERGENCE OF FINITE-VOLUME MPDATA ON UNSTRUCTURED SKEWED MESH

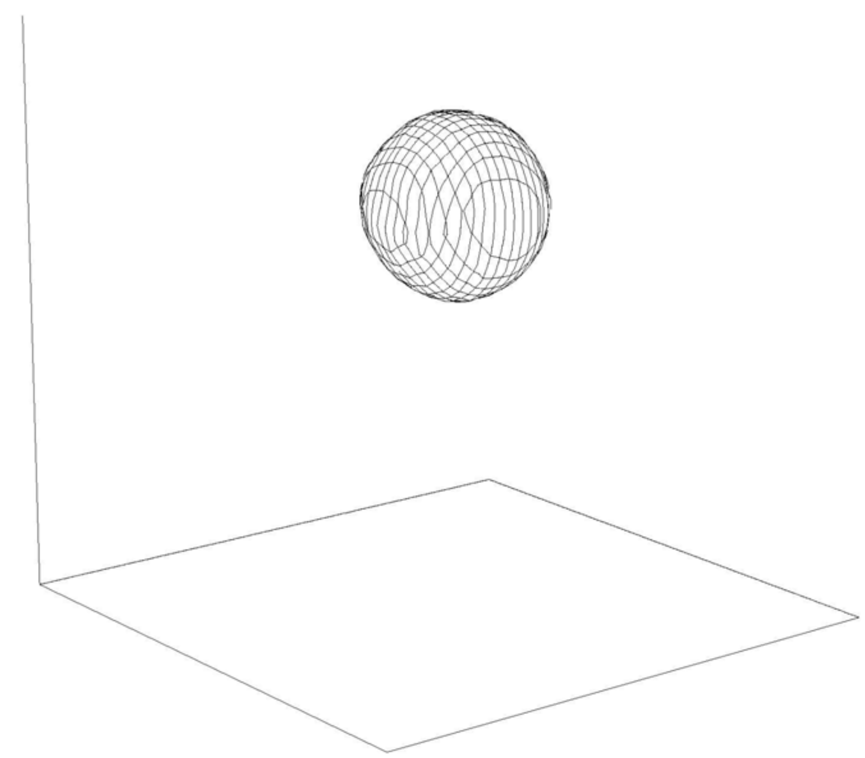


δ_E	L_∞	L_2	L_1
3.99	$3.58{\cdot}10^{-1}$	$1.38{\cdot}10^{-1}$	$2.86{\cdot}10^{-2}$
2.05	$2.01{\cdot}10^{-1}$	$6.68{\cdot}10^{-2}$	$1.38{\cdot}10^{-2}$
1.03	$8.51{\cdot}10^{-2}$	$2.34{\cdot}10^{-2}$	$4.78{\cdot}10^{-3}$
0.51	$1.86{\cdot}10^{-2}$	$5.14{\cdot}10^{-3}$	$9.41{\cdot}10^{-4}$
0.26	$3.23{\cdot}10^{-2}$	$1.48{\cdot}10^{-3}$	$2.52{\cdot}10^{-4}$

REVOLUTION OF A SPHERE AROUND THE DIAGONAL OF A DOMAIN



INITIAL



MPDATA GAGE AFTER 1 REVOLUTION

$$\frac{\partial \Phi}{\partial t} + \nabla \bullet (\mathbf{V}\Phi) = \mathbf{R}$$

NFT MPDATA: a general template

$$\Phi_i^{n+1} = \Phi_i^* + 0.5\delta t \mathbf{R}_i^{n+1}$$

$$\Phi^* \equiv \mathcal{A}(\Phi^n + 0.5\delta t \mathbf{R}^n, \widehat{\mathbf{V}}^{n+1/2})$$

$$R^{n+1} = R(t + \delta t) + \mathcal{O}(\delta t^3)$$

implicit integration

preconditioned non-symmetric Krylov-subspace elliptic solver

explicit integration

$$\forall_i \quad \Phi_i^{n+1,\ \mu} = \Phi_i^* + 0.5\delta t \mathbf{R}_i^{n+1,\ \mu-1} \qquad \mu = 1, .., m$$

Explicit integration

Compressible flows

(Smolarkiewicz, Szmelter JCP 2008 published on line)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\mathbf{V} \rho) = 0$$

$$\frac{\partial q^I}{\partial t} + \nabla \bullet (\mathbf{V} q^I) = -\frac{\partial p}{\partial x^I} \, .$$

$$\frac{\partial e}{\partial t} + \nabla \bullet (\mathbf{V} e) = -p \nabla \bullet \mathbf{V}$$

Internal Energy Equation

$$\frac{\partial \Theta}{\partial t} + \nabla \bullet (\mathbf{V} \Theta) = 0$$

Potential Temperature Equation

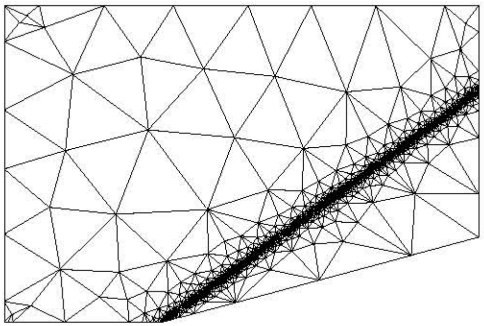
$$\frac{\partial E}{\partial t} + \nabla \bullet (\mathbf{V} E) = -\nabla \bullet (\mathbf{V} p)$$

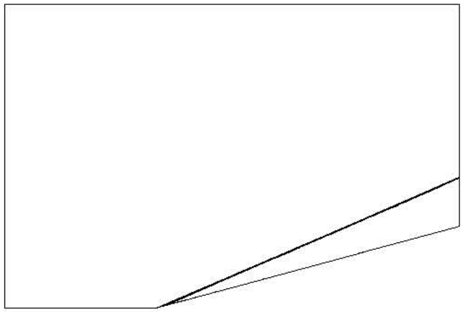
Classical Total Energy Equation

$$\frac{\partial E}{\partial t} + \nabla \bullet (\widetilde{\mathbf{V}} E) = 0 \qquad \widetilde{\mathbf{V}} \equiv \mathbf{V}(1 + p/E)$$

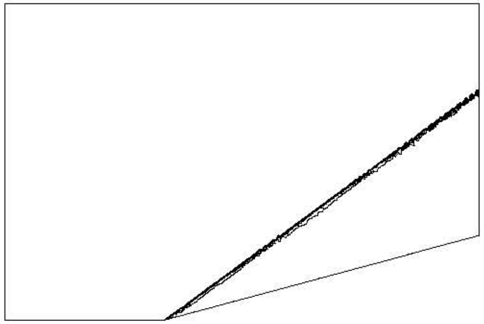
Modified Total Energy Equation

Mesh adaptivity by remeshing

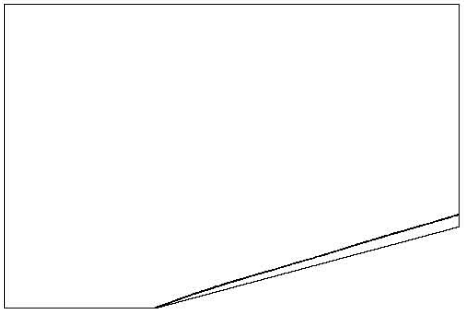




M = 5



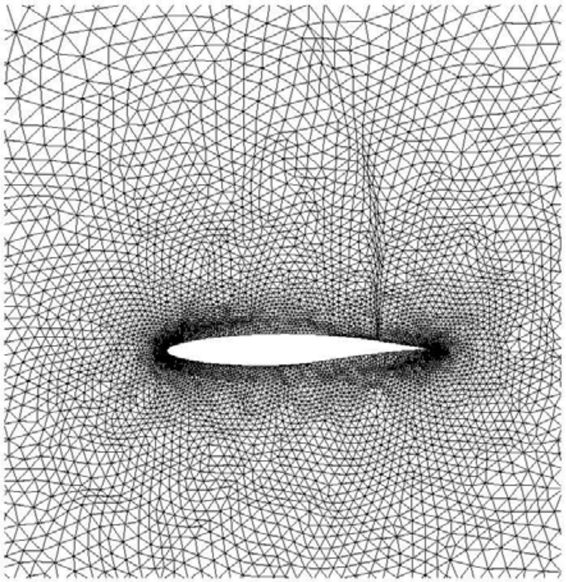
M =2. 5



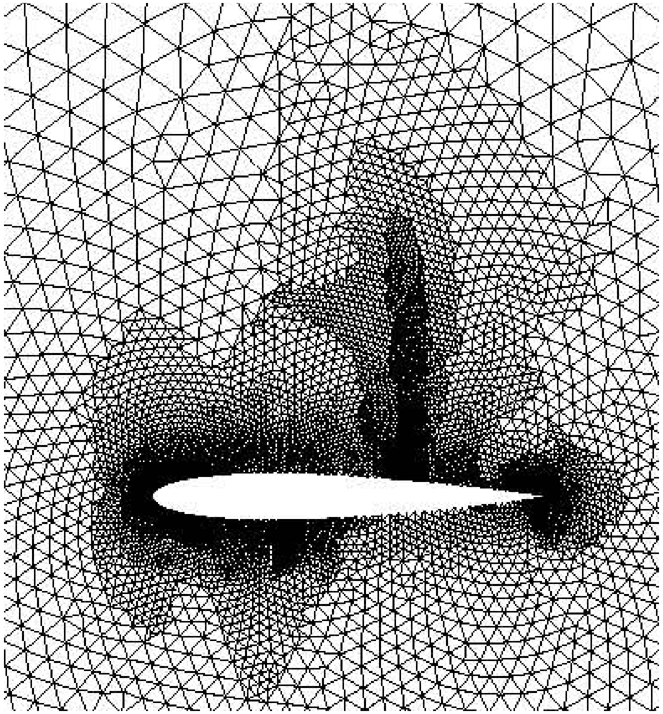
M = 15

ADAPTIVITY with a refinement indicator based on MPDATA lead error

(Szmelter,Smolarkiewicz IJNMF 2005)



Mesh movement



Point enrichment

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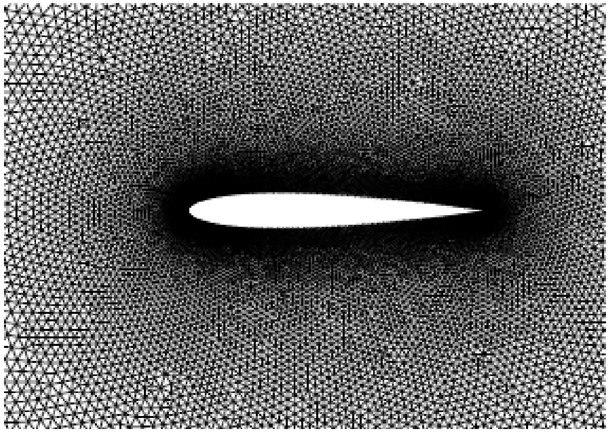
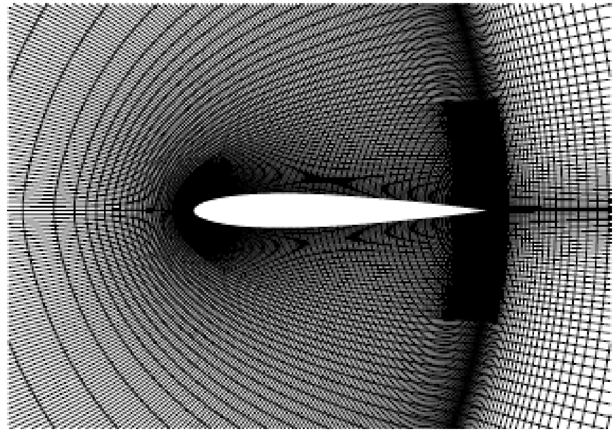
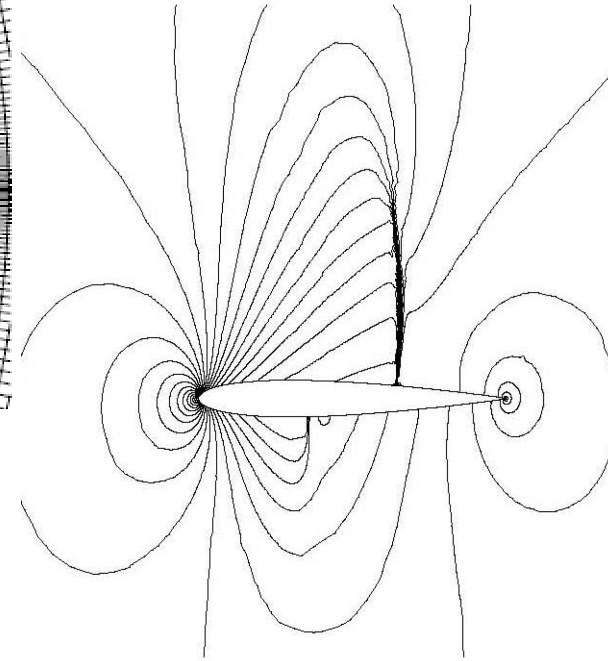
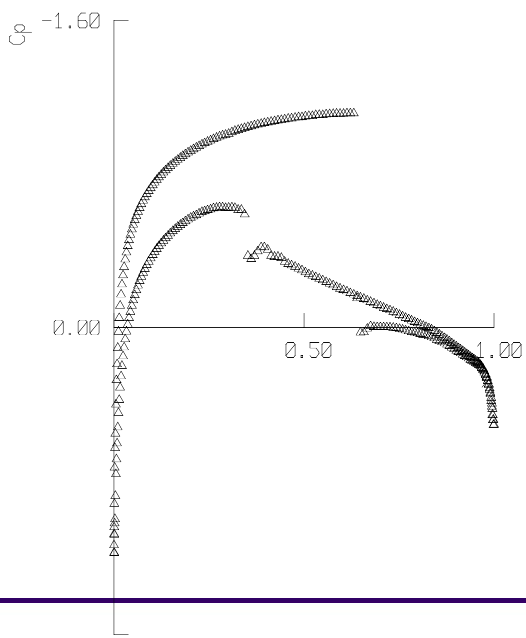
Transonic flow

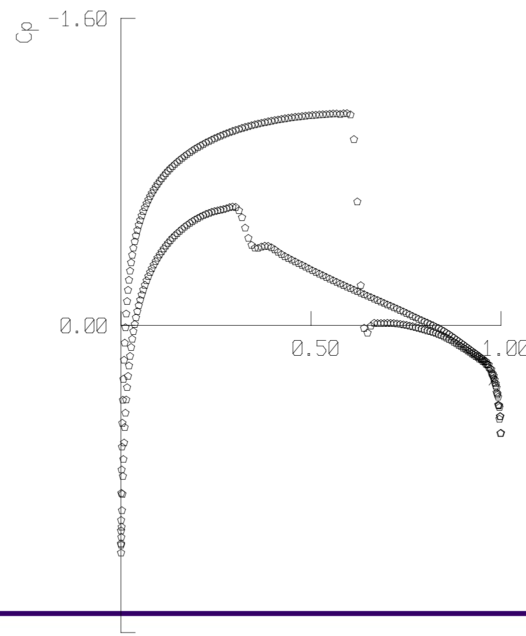
NACA012 M=0.85 α=1.25º











MPDATA

Runge-Kutta

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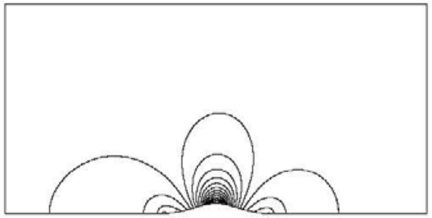

CONVERGENCE STUDY

MPDATA - NFT EULER SOLVER

M=0.5

δ_E	MPDATA L_2	UPWIND L_2	L_1	
$4.47\cdot10^{-2}$	$4.87\cdot10^{-3}$	$(5.47\cdot10^{-3})$	$1.48\cdot10^{-3}$	$(1.97\cdot10^{-3})$
$3.10\cdot10^{-2}$	$3.58\cdot10^{-3}$	$(4.18\cdot10^{-3})$	$1.19\cdot10^{-3}$	$(1.73\cdot10^{-3})$
$2.19\cdot10^{-2}$	$1.64\cdot10^{-3}$	$(2.83\cdot10^{-3})$	$7.04\cdot10^{-4}$	$(1.51\cdot10^{-3})$
$1.54\cdot10^{-2}$	$1.01\cdot10^{-3}$	$(2.17\cdot10^{-3})$	$4.36\cdot10^{-4}$	$(1.21\cdot10^{-3})$
$1.07\cdot10^{-2}$	$6.24\cdot10^{-4}$	$(1.58\cdot10^{-3})$	$2.38\cdot10^{-4}$	$(9.01\cdot10^{-4})$
$7.62\cdot10^{-3}$	$3.38\cdot10^{-4}$	$(1.19\cdot10^{-3})$	$1.25\cdot10^{-4}$	$(6.90\cdot10^{-4})$
$5.35\cdot10^{-3}$	$1.77\cdot10^{-4}$	$(8.57\cdot10^{-4})$	$6.61\cdot10^{-5}$	$(5.10\cdot10^{-4})$
$3.78\cdot10^{-3}$	$8.50\cdot10^{-5}$	$(6.06\cdot10^{-4})$	$2.92\cdot10^{-5}$	$(3.65\cdot10^{-4})$
$2.67\cdot10^{-3}$	$4.66\cdot10^{-5}$	$(4.34\cdot10^{-4})$	$1.79\cdot10^{-5}$	$(2.64\cdot10^{-4})$

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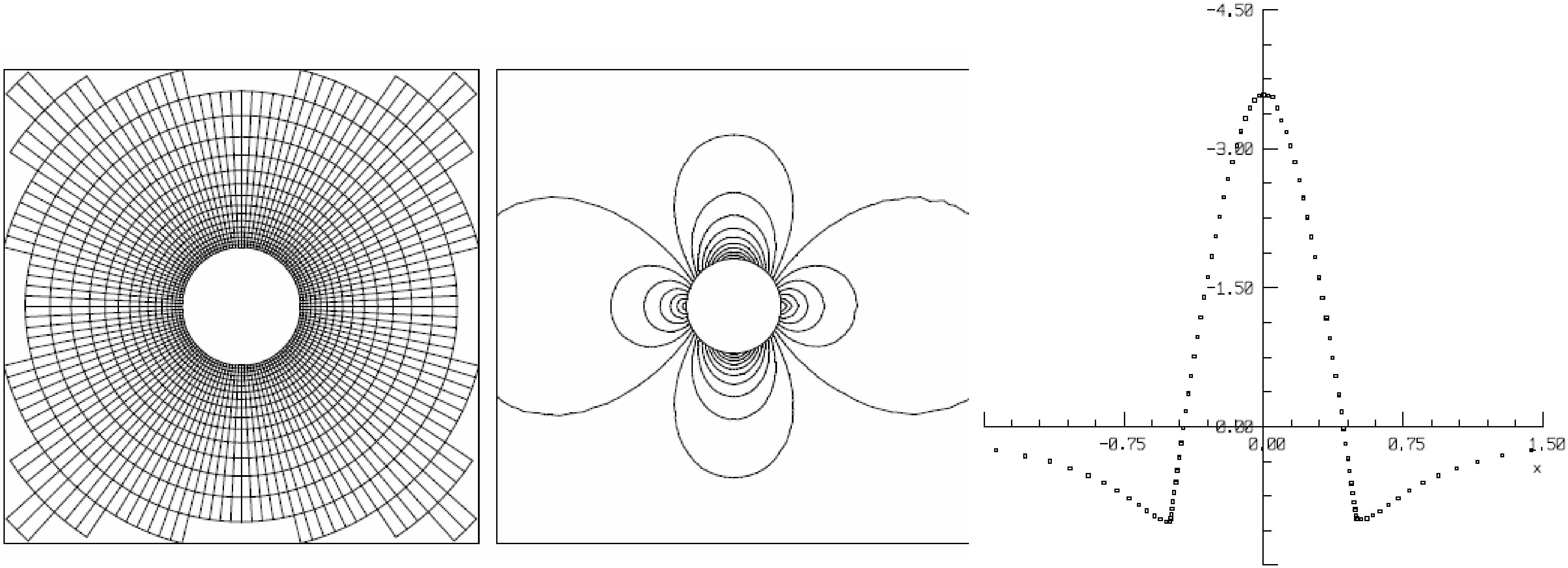
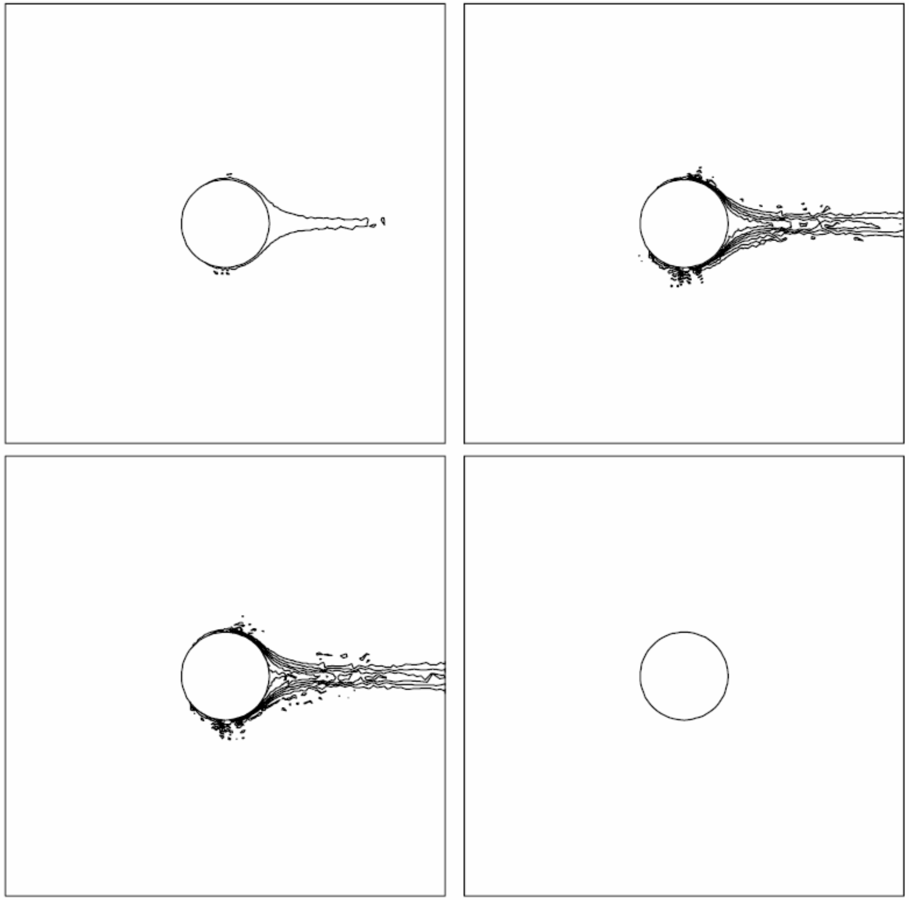
Flow past a cylinder M=0.38



Flow past a cylinder M=0.38 --- Entropy deviation triangular meshes

Internal energy

Total energy



Modified total energy

Potential temperature

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Flow past a cylinder

Equations; MPDATA	Mach (Min,Max)	Pressure (Min,Max)	Σ (Min,Max)
	Structured 128×33 O-grid		
internal energy (16)	0.00000,0.91	0.64,1.11	-0.00002,0.00021
total energy (17)	0.00000,0.91	0.62,1.11	-0.00005,0.00081
total energy (18)	0.00000,0.91	0.63,1.11	-0.00022,0.00125
potential temp. (19)	0.00000,0.91	0.62,1.11	-0.00000,0.00001
	Unstructured triangular mesh		
internal energy (16)	0.00001,0.90	0.64,1.11	-0.00037,0.00053
total energy (17)	0.00000,0.90	0.63,1.11	-0.00098,0.00247
total energy (18)	0.00000,0.90	0.63,1.11	-0.00130,0.00406
potential temp. (19)	0.00000,0.90	0.62,1.11	-0.00005,0.00005

Scheme; after [52]	Mach (Min,Max)	Pressure (Min,Max)	Σ (Min,Max)
MUSCL	0.0001 ,0.82	0.67,1.10	-0.0004 ,0.048
Abgrall's blended	0.0001 ,0.89	0.64,1.11	0.0 ,0.009
LDA	0.0 ,0.94	0.62,1.10	-0.001 ,0.010

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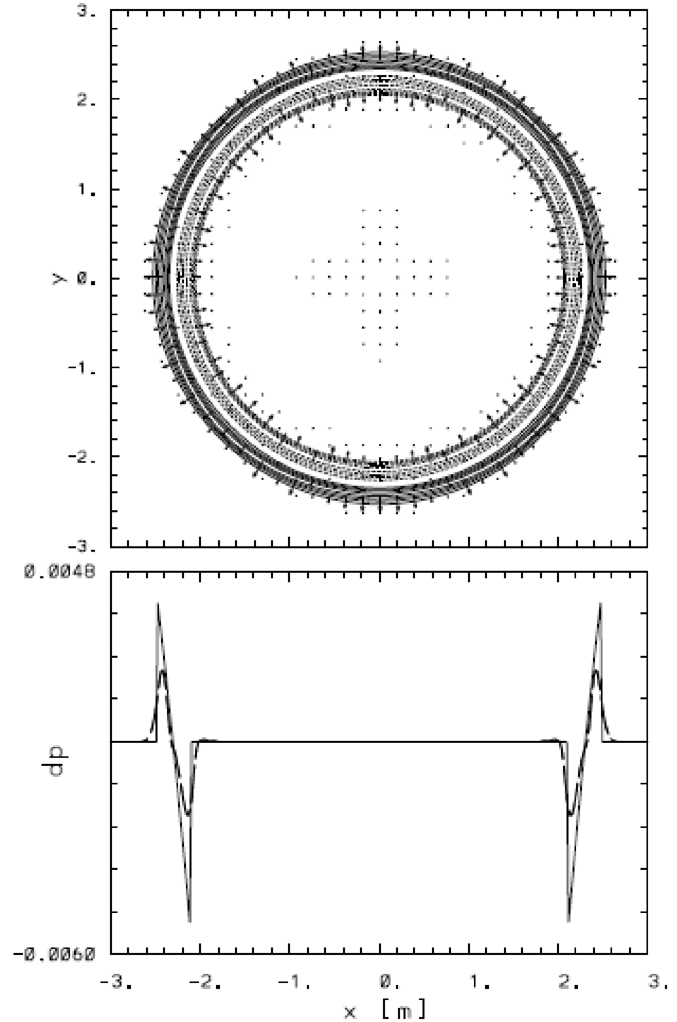

M=0

$$\delta p = (p - p_o)/p_o \text{ at } y = z = 0$$

$$t \gg r_o/c_o$$

3D Spherical wave; Analytical solution

--- Landau & Lifshitz

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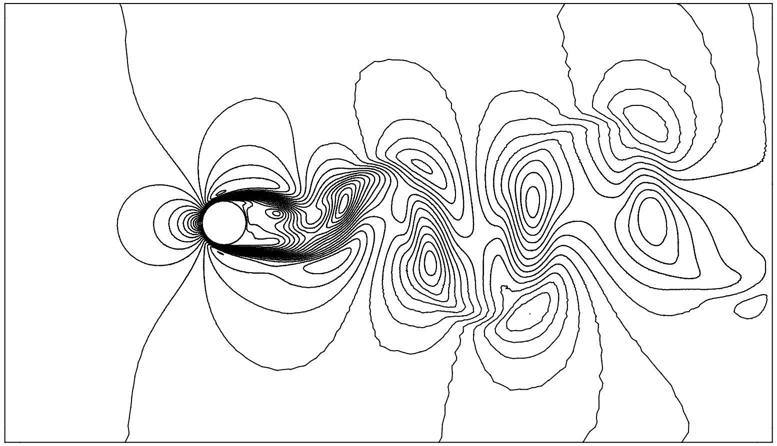
Implicit integration

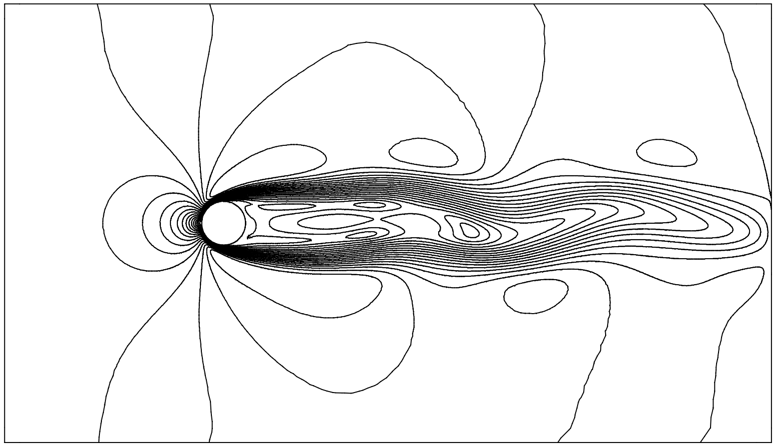
Incompressible flows

Incompressible flows



Re=200.

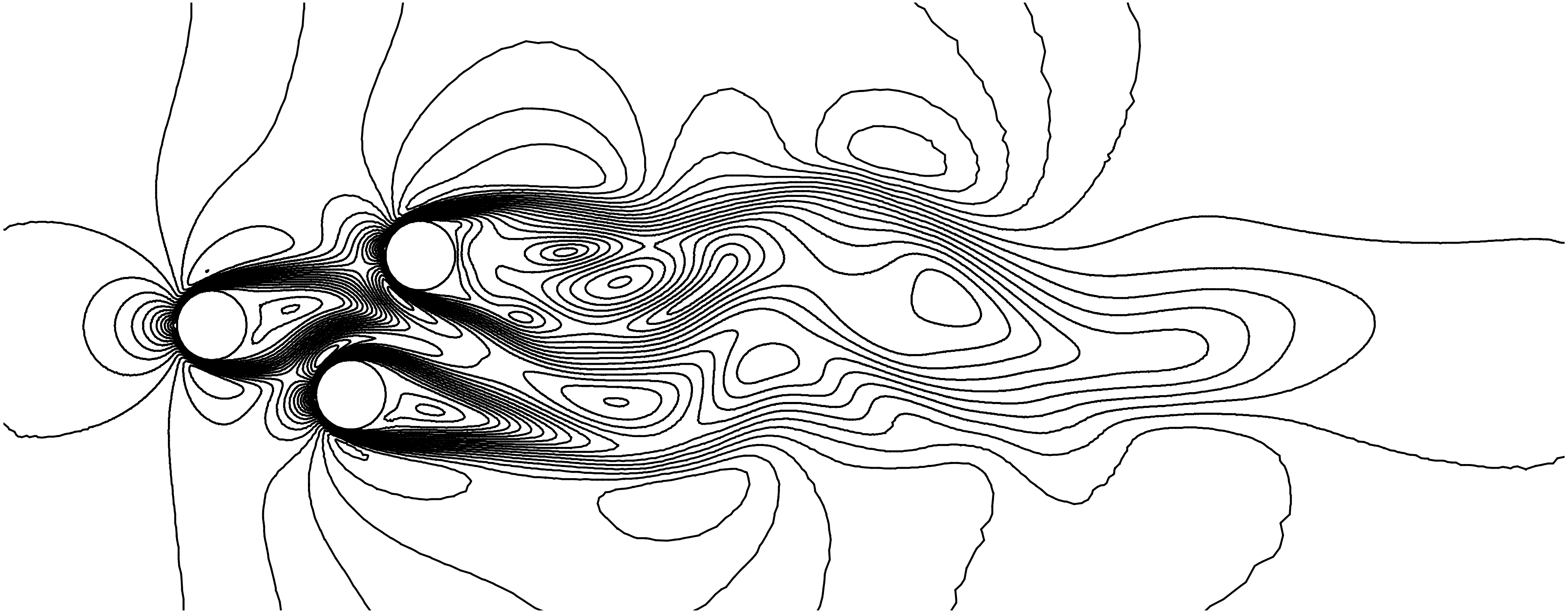
Strouhal number =183



Re=100.

Strouhal number =163

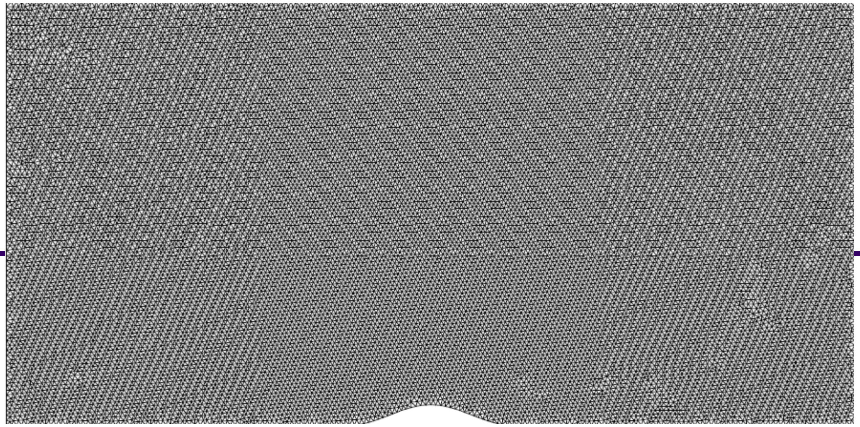
Incompressible flows --- multi-connected domains



Re=100

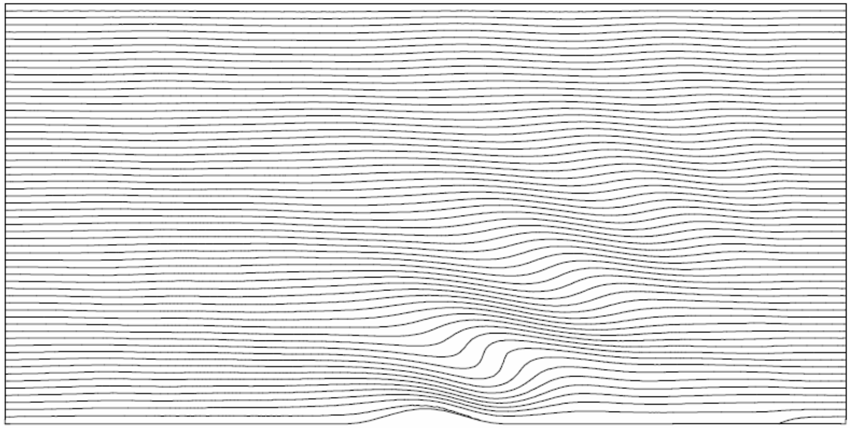
2D – non-hydrostatic model
Incompressible Boussinesq
ILES

Orographically forced atmospheric gravity waves



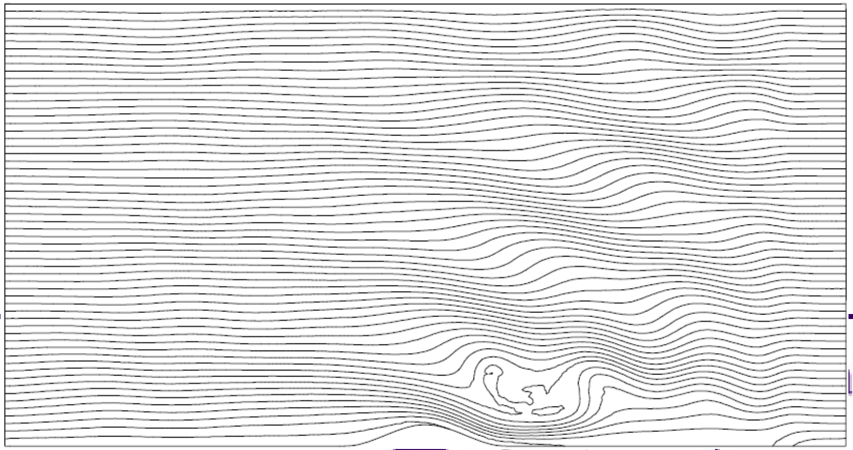
$$Fr < Fr_{critical}$$

vertical wave propagation

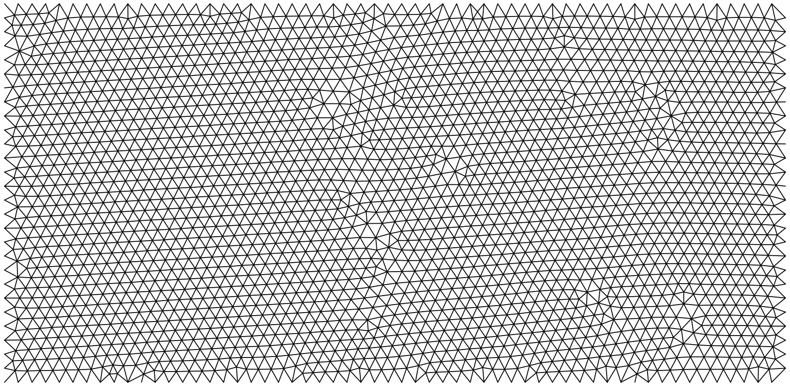


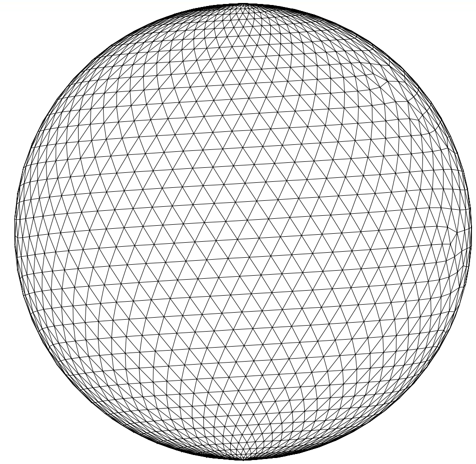
$$Fr > Fr_{critical}$$

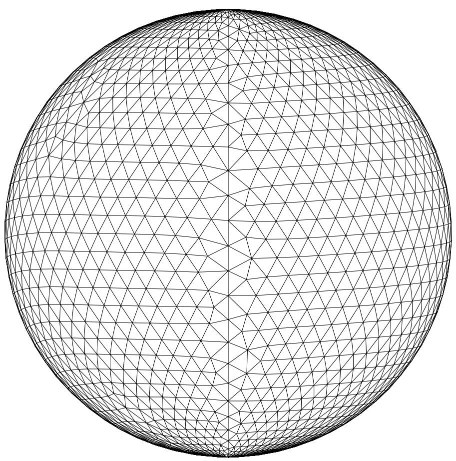
wave breaking

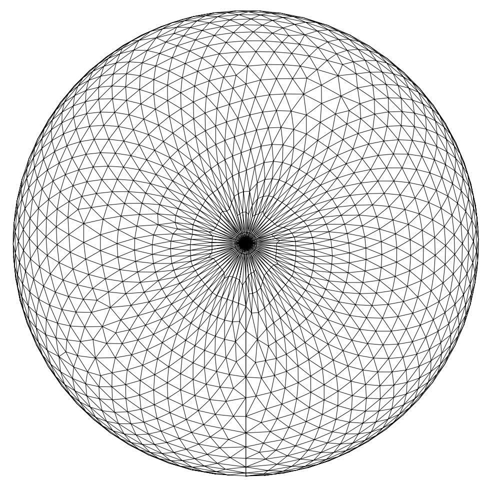


Flows on a Sphere --- Geospherical coordinates

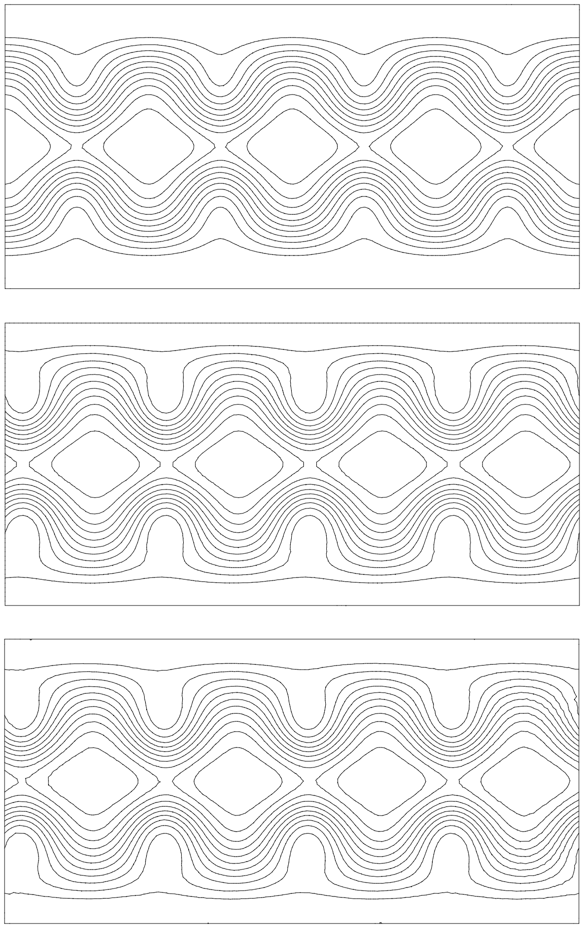




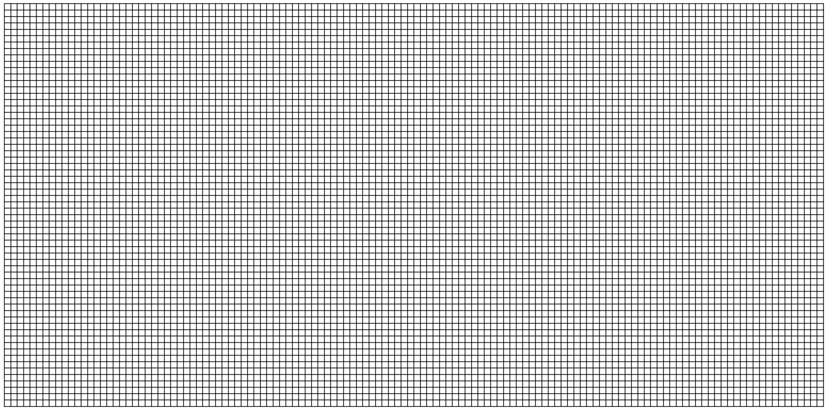


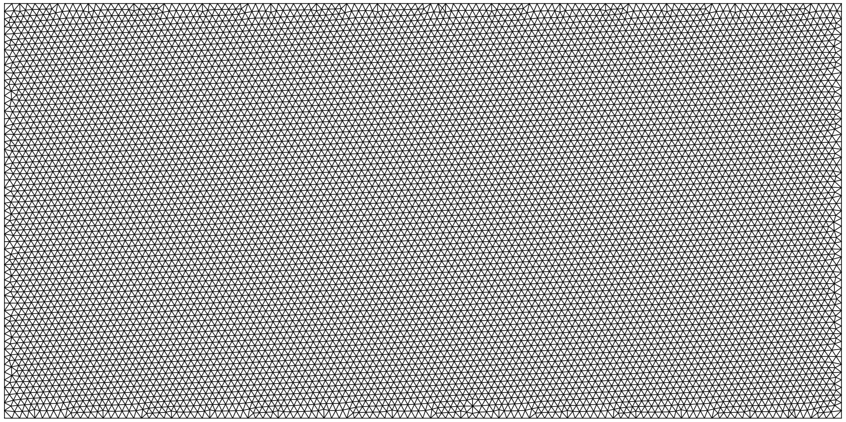


Shallow water equations

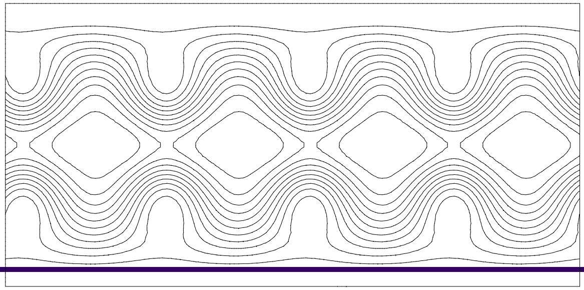


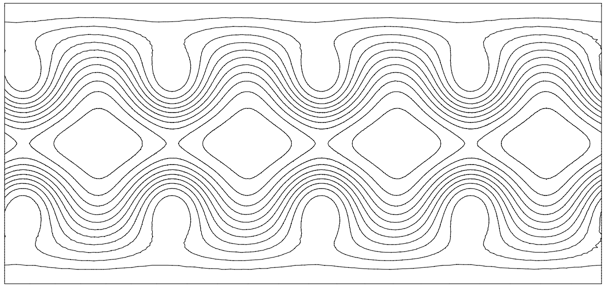
initial

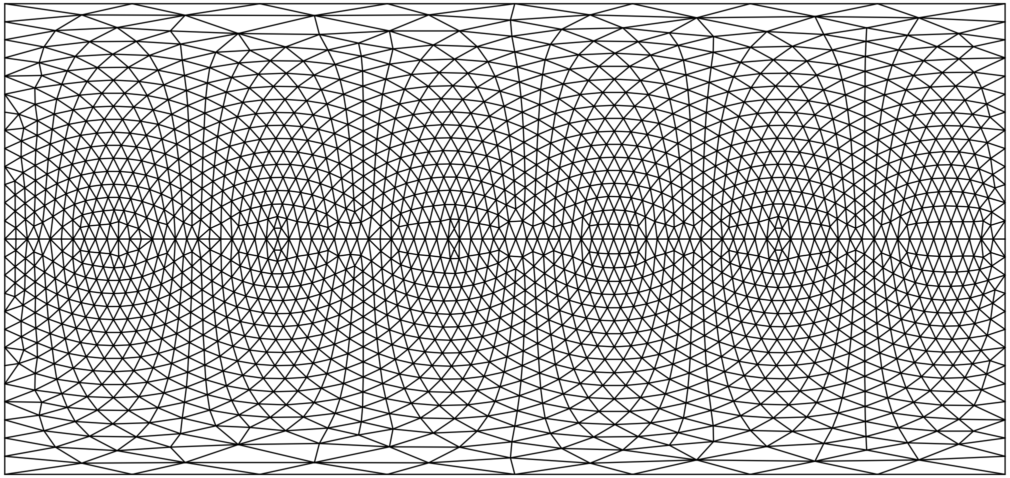


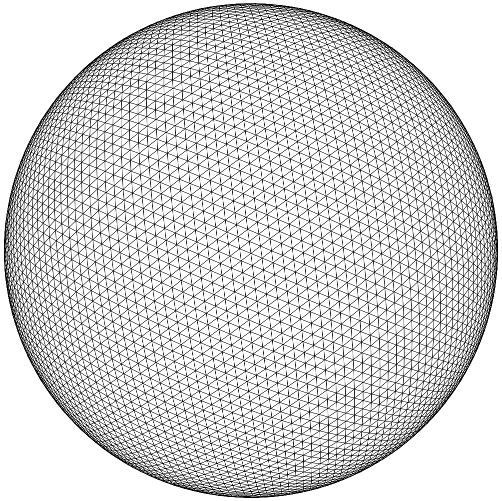


Shallow water equations









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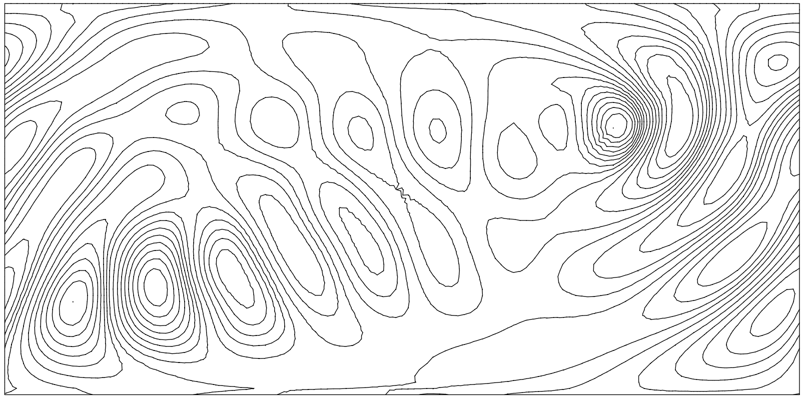
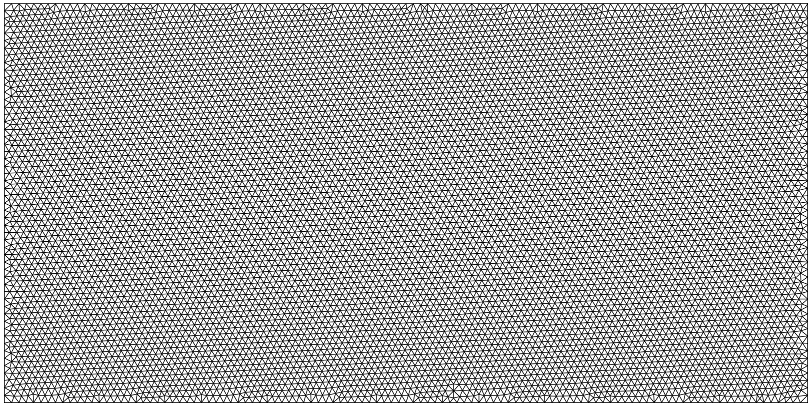
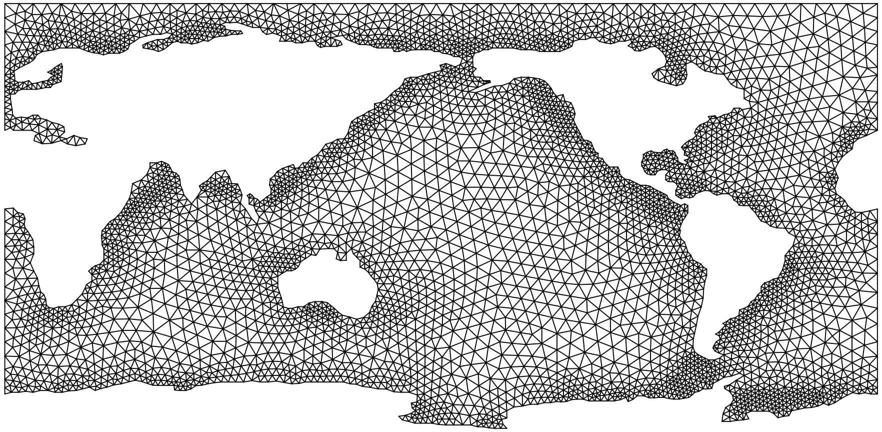
Shallow water equations

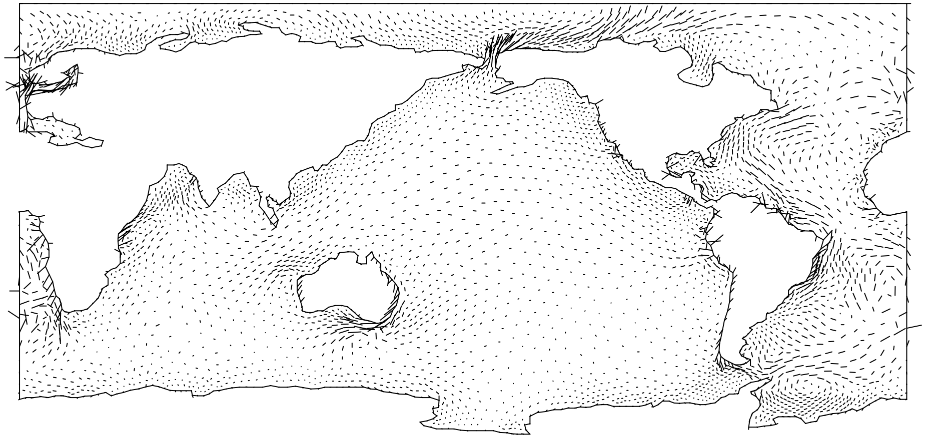


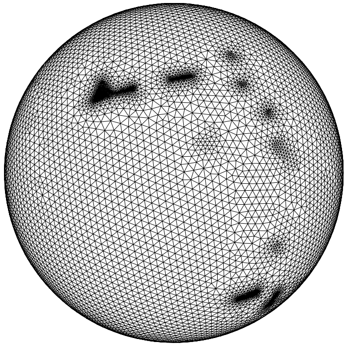


Zonal flow over a cone

Potential for modelling of multi-connected domains and easy implementation of mesh adaptivity. (*David Parkinson*)







(*Treatment of hanging nodes & hybrid meshes*

Szmelter et al Comp. Meth. Appl. Mech. Eng 92)

Loughborough University

CONCLUSIONS

- Presented work opens avenues to construct a flexible edge-based clone of EULAG and to study unstructured meshes performance for all-scale atmospheric flows.
- Main new effort lies in mesh generation, visualisation and parallelisation.