ADAPTIVE GRID TRANSFORMATIONS AND GENERALIZED COORDINATES IN EULAG

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Turbulence is ubiquitous…

(Van Dyke, Parabolic P.1982)

Re=2300; water jet in water

$L / \lambda_k \sim Re^{3/4} \sim 330$

λ_k is Kolmogorov microscale

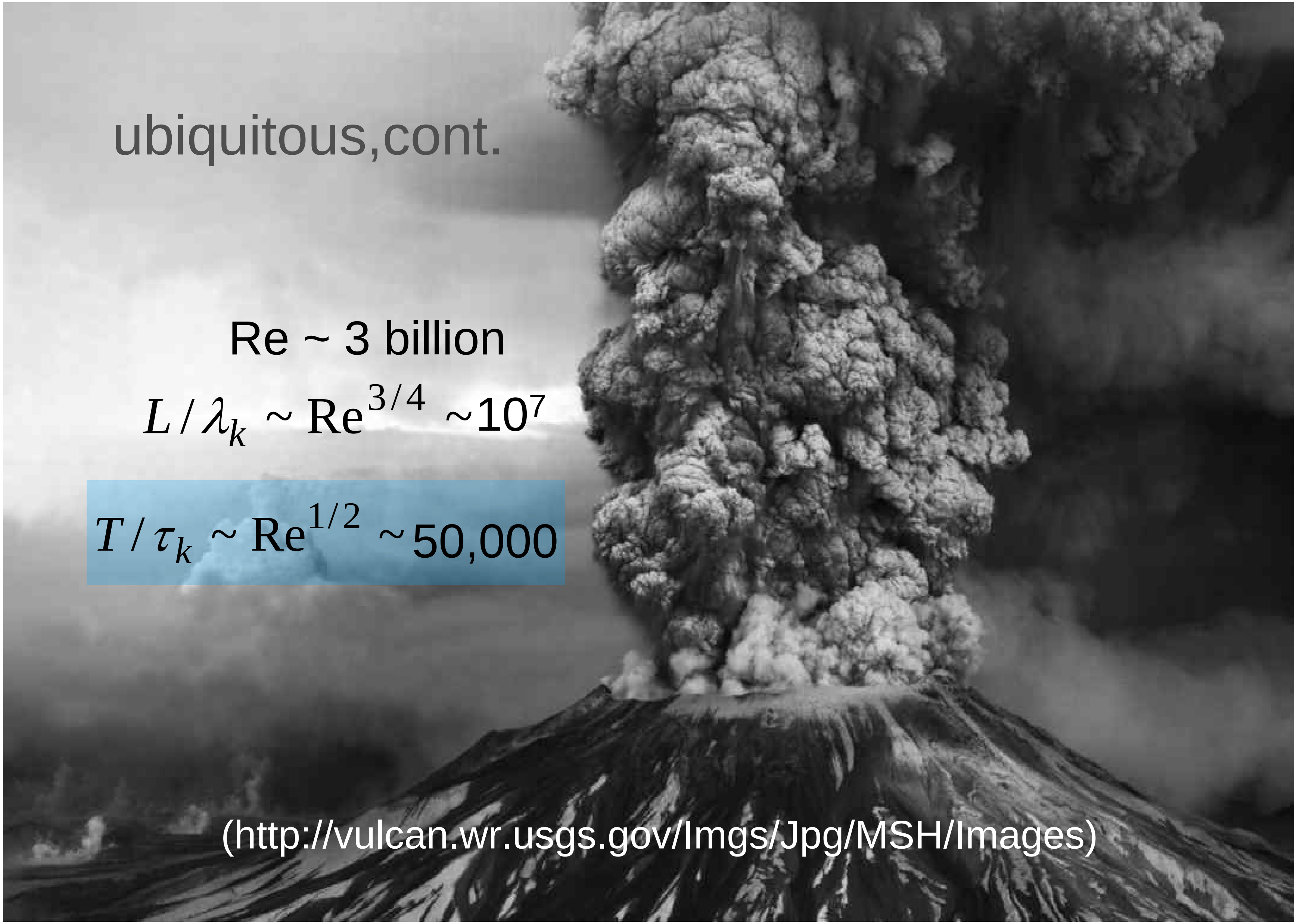
(Tennekes and Lumley, MIT P. 1972)

ubiquitous,cont.

Re ~ 3 billion

$$L / \lambda_k \sim \mathrm{Re}^{3/4} \sim 10^7$$

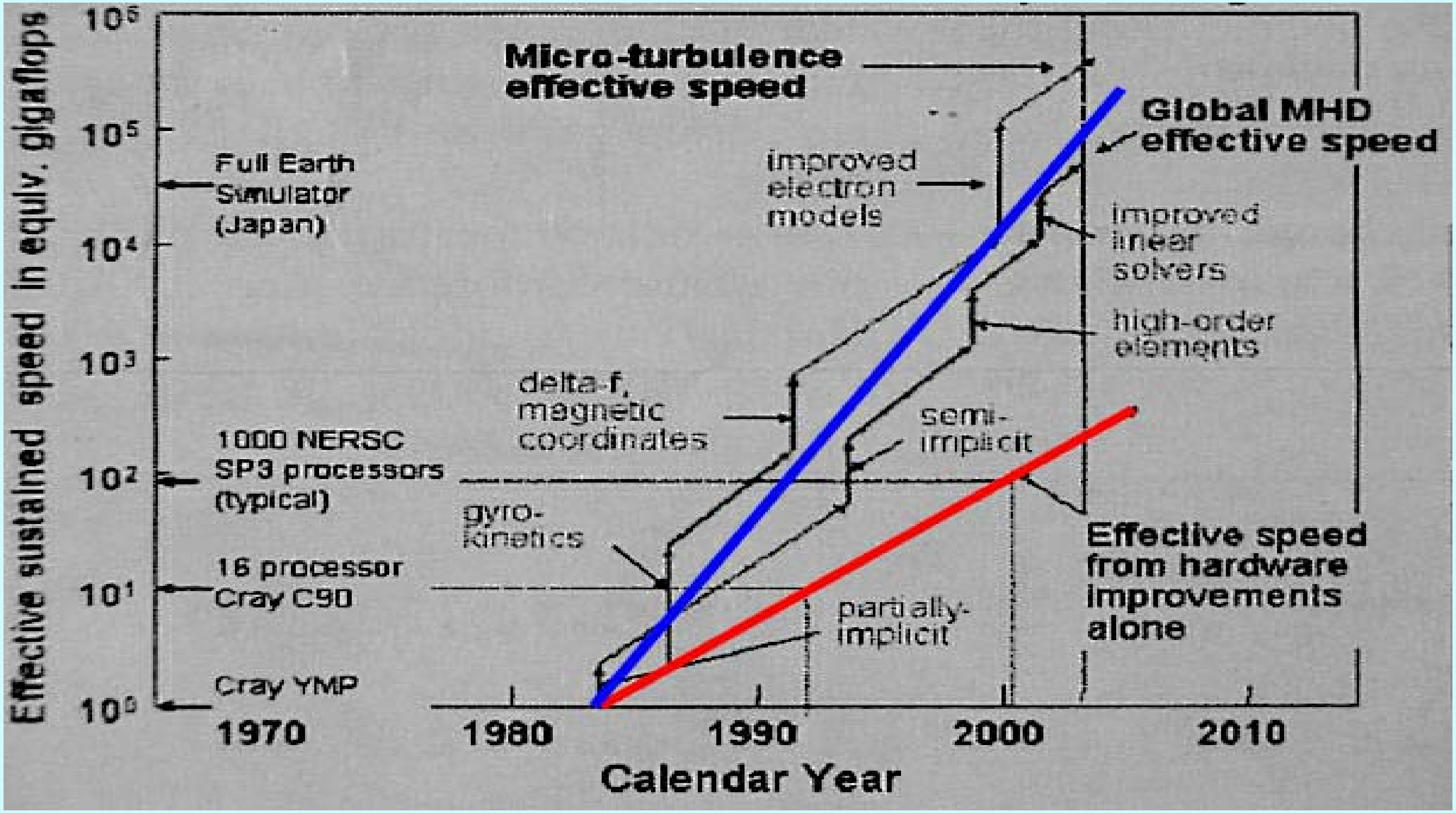
$$T / \tau_k \sim \mathrm{Re}^{1/2} \sim 50{,}000$$



(http://vulcan.wr.usgs.gov/Imgs/Jpg/MSH/Images)

Extended “Moore’s Law” for MHD simulations

(Crowley, *SIAM News* 2004)

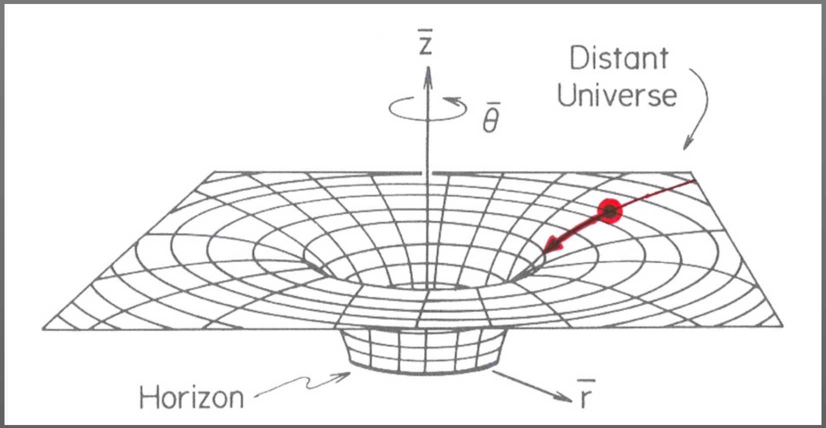


GA in Computational Models, cont.

In fusion simulations, it is anticipated that GA will contribute to model speed-up more than hardware improvements (Sipics 2006 SIAM News)

GA in Computational Models, cont.

- Consider an example from general relativity



A test particle free falls into a black hole:

1. From the particle's point of view, *proper time* τ, it's passage through the horizon is swift.

$$V = \sqrt{2M/r}$$

2. From the distant universe at rest, coordinate time *t*, the particle appears to approach the horizon asymtotically, and NEVER reaches it in finite time.

$$V = (1-2M/r)\sqrt{2M/r} \approx \Delta r/(2M)$$

"Time is defined so that motion looks simple" (Misner et.al, Freeman P. 1973).

$$d\tau = (1-2M/r)^{1/2}\, dt$$

GA in Computational Models, concluded

BEST COORDINATES:

- make dynamics look "simple"by redefining space
- think inner and outer scales in spirit of perturbation theory (Holmes, *Springer-Verlag P.* 1995)

- GA attempts to approximate best coordinates

(i) reduces computational resources needed for a given resolution ⟷ resolves smaller scales for a given computational resource

(ii) solution better reveals relevant physics

Consider GA an *essential feature*

GA in EULAG

Introduce coordinate transformation

(Prusa & Smolarkiewicz, *JCP* 2003)

$$(\bar{t},\bar{x},\bar{y},\bar{z}) = \big(t, E(t,x,y), D(t,x,y), C(t,x,y,z)\big)$$

where $(t,x,y,z) \in \mathbf{D}_p \subseteq \mathbf{S}_p$: physical space

and $(\bar{t},\bar{x},\bar{y},\bar{z}) \in \mathbf{D}_t \subseteq \mathbf{S}_t$: transformed space

$\mathbf{D}_p, \mathbf{D}_t$ are physical and transformed computational domains

- Physical problem is posed in physical space, $\mathbf{S}_p$. The coordinates $(t,\mathbf{x})$ describing $\mathbf{S}_p$ are *stationary* and *orthogonal*
- Physical problem is solved in transformed space, $\mathbf{S}_t$. The coordinates $(\bar{t},\bar{\mathbf{x}})$ describing $\mathbf{S}_t$ are nonstationary and nonorthogonal as viewed from $\mathbf{S}_p$

EULAG over., cont.

$$\frac{\partial(\rho^* \overline{v^s}^k)}{\partial \overline{x}^k} = 0 \ .$$

$$\frac{dv^j}{d\bar{t}} = -\ \widetilde{G}_j^k \frac{\partial \pi'}{\partial \overline{x}^k} + g\frac{\theta'}{\theta_b}\delta_3{}^j + \mathcal{F}^j + \mathcal{V}^j \ ,$$

$$\frac{d\theta'}{d\bar{t}} = -\overline{v^s}^k \frac{\partial \theta_e}{\partial \overline{x}^k} + \mathcal{H} \ ,$$

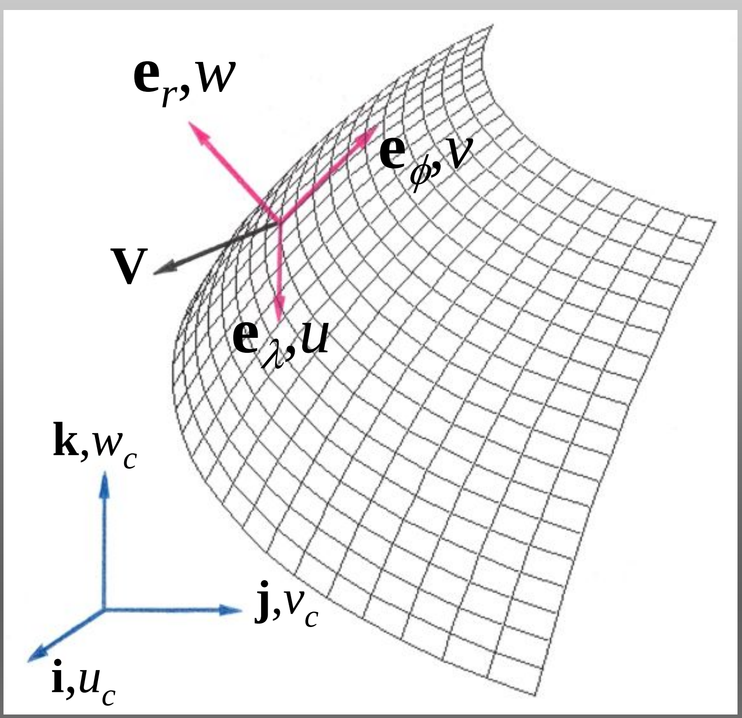
- 3 velocities: v^j ("physical" as described in $\mathbf{S}_p$)

  $\overline{v}^{s^k}$ ("solenoidal" as described in $\mathbf{S}_t$)

  $\overline{v}^{*^k}$ ("contravariant" = advection as described in $\mathbf{S}_t$)
- Metric coefficients: $\tilde{G}_j^k := \sqrt{g^{jj}}\left(\partial\overline{x}^k / \partial x^j\right)$

EULAG velocities, cont.

- Velocities for S_p spherical coordinates



physical: $\mathbf{V} = u\mathbf{e}_\lambda + v\mathbf{e}_\phi + w\mathbf{e}_r$

$= u_c\mathbf{i} + v_c\mathbf{j} + w_c\mathbf{k}$

$$u = -u_c \sin\lambda + v_c \cos\lambda \quad (zonal)$$

$$v = -u_c \sin\phi \cos\lambda - v_c \sin\phi \sin\lambda + w_c \cos\phi \quad (meridional)$$

$$w = u_c \cos\phi \cos\lambda + v_c \cos\phi \sin\lambda + w_c \sin\phi \quad (radial)$$

contravariant: $u^* = u/(\Gamma \cos\phi), \quad v^* = v/\Gamma, \quad w^* = w$

(where $\Gamma = 1 + z/R_o$, $\lambda R_o = x$, $\phi R_o = y$, $z = r - R_o$)

EULAG velocities, cont.

Contravariant $\bar{v}^{*k}$ is analytically most fundamental form

$$\bar{v}^{*i} := \frac{d\bar{x}^i}{d\bar{t}} = v^{*i}\,\frac{\partial \bar{x}^i}{\partial x^j}$$

grid speed

$$= \frac{\partial \bar{x}^i}{\partial t} + \left(u^* \frac{\partial \bar{x}^i}{\partial x} + v^* \frac{\partial \bar{x}^i}{\partial y} + w^* \frac{\partial \bar{x}^i}{\partial z} \right) = \boxed{\frac{\partial \bar{x}^i}{\partial t}} + \bar{v}^{s^i}$$

Solenoidal $\bar{v}^{s^k}$ is convenient for anelastic continuity

Physical $v^k = \sqrt{g_{kk}}\,v^{*k}$ is most easily measured

EULAG metrics, cont.

- Metric tensors for $\mathbf{S}_p$: $g_{kk} = \delta_{kk}$ (Cartesian)

  g_{jk}=0 for $j \neq k$

$$g_{kk} = \delta_{k1} + \delta_{k2}\Gamma^2 + \delta_{k3}$$ (polar cylindrical, $\Gamma = r / R_o$)

$$g_{kk} = \delta_{k1}(\Gamma \cos\phi)^2 + \delta_{k2}\Gamma^2 + \delta_{k3}$$ (spherical)

Conjugate metric tensors for $\mathbf{S}_p$: $g^{kk} = g_{kk}^{-1}$

Conjugate metric tensor for $\mathbf{S}_t$:

$$\overline{g}^{mn} = g^{kk}\left(\frac{\partial \overline{x}^m}{\partial x^k}\frac{\partial \overline{x}^n}{\partial x^k}\right) \rightarrow \overline{g}^{12} = \left(D_{,y}E_{,y} + D_{,x}E_{,x} / \cos^2\phi\right)\Gamma^{-2}, \quad \ldots$$

EULAG metrics, cont.

- Jacobians for S$_p$, $G = |g_{jk}|^{1/2}$:

  G=1 (Cartesian), Γ (polar cylindrical), and

  $\Gamma^2\cos\phi$ (spherical coordinates)

Jacobian for S$_t$ is separable:

$$\overline{G} = G\,\overline{G'} = G\,(\overline{G}_o\overline{G}_{xy})$$

- □Vertical mapping contribution

$$\overline{G}_o = \left(\frac{H(t,x,y) - z_s(t,x,y)}{H_o}\right)\left(\frac{dC}{d\xi}\right)^{-1}$$

EXTENDED GAL-CHEN
(Wedi & Smolarkiwicz, *JCP* 2004)

EULAG metrics, cont.

where $$\bar{z} = C(\xi);\ \ \xi = H_o\left(\frac{z - z_s}{H - z_s}\right)$$ is a similarity variable that collapses the dependency of $C(t,x,y,z)$ onto that of a single independent variable

- Horizontal mapping:

$$\overline{G}_{xy} = \left(\mathrm{E}_{,\bar{x}}\mathrm{D}_{,\bar{y}} - \mathrm{D}_{,\bar{x}}\mathrm{E}_{,\bar{y}}\right) = \left(E_{,x}D_{,y} - D_{,x}E_{,y}\right)^{-1}$$

where $(\mathrm{E},\ \mathrm{D})$ are the mappings

$$(x,y) = \left((\mathrm{E}(\bar{t},\bar{x},\bar{y}),\mathrm{D}(\bar{t},\bar{x},\bar{y})\right)$$

Metric identities

1. Kronecker Delta (KD) identities:

   (Prusa and Gutowski, *IJNMF* 2006)

$$\frac{\partial \bar{x}^m}{\partial \bar{x}^k} = \bar{\delta}_k^m, \quad \bar{x}^m = \bar{x}^m(x^j) \quad \rightarrow \quad \frac{\partial \bar{x}^m}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^k} = \bar{\delta}_k^m$$

Why are these important? Consider computation of the contravariant velocity:

$$\bar{v}^{*m} = v^{*k}\frac{\partial \bar{x}^m}{\partial x^k} \rightarrow \bar{v}^{*m}\frac{\partial x^j}{\partial \bar{x}^m} = v^{*k}\left(\frac{\partial \bar{x}^m}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^m}\right) \equiv v^{*k}\delta_k^j$$

$$\rightarrow \quad \bar{v}^{*m}\frac{\partial x^k}{\partial \bar{x}^m} = v^{*k}$$

Use of the KD identities is ubiquitous in tensor manipulations

identities, cont.

How are KD indentities implemented?

Consider the case m = 1, k = 0:

$$\frac{\partial \bar{x}^1}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^0} = \bar{\delta}_0^1 \quad \rightarrow \quad \frac{\partial \bar{x}}{\partial x^j}\frac{\partial x^j}{\partial \bar{t}} = 0$$

↓ zero since $x = E(t,x,y)$

$$\rightarrow \quad E_{,t} + E_{,x}\mathrm{E}_{,\bar{t}} + E_{,y}\mathrm{D}_{,\bar{t}} + E_{,z}\mathrm{C}_{,\bar{t}} = 0$$

case m = 1, k = 1: $E_{,x}\mathrm{E}_{,\bar{x}} + E_{,y}\mathrm{D}_{,\bar{x}} = 1$, …

In general, there are 16 independent KD equations; but given the allowed form of the mapping, only 10 are nontrivial

Solve for $\partial \bar{x}^m / \partial x^j$: $E_{,x} = \mathrm{D}_{,\bar{y}} / G_{xy}$

$$E_{,t} = \left(\mathrm{E}_{,\bar{y}}\mathrm{D}_{,\bar{t}} - \mathrm{D}_{,\bar{y}}\mathrm{E}_{,\bar{t}}\right)/G_{xy}$$, …

EULAG metrics, cont.

2. Geometric Conservation Law (GCL) identities: (Prusa and Gutowski, *IJNMF* 2006)

Motivation? Consider the computation of the divergence of a contravariant vector F^j (e.g., some physical flux $F^j = \alpha g^{jk}(\partial f / \partial x^k)$)

$$\nabla \bullet \mathbf{F} \equiv \frac{1}{G}\frac{\partial}{\partial x^j}\left(G F^j\right)$$

Tensor form in physical coordinates

$$\rightarrow \quad \overline{\nabla} \bullet \overline{\mathbf{F}} \equiv \frac{1}{\overline{G}}\frac{\partial}{\partial \overline{x}^j}\left(\overline{G}\,\overline{F}^j\right)$$

Tensor form in transformed coordinates

Is anything more required for tensor character to be preserved?

EULAG metrics, cont.

$$\nabla \bullet \mathbf{F} \equiv \frac{1}{G}\frac{\partial}{\partial x^j}\left(GF^j\right) = \frac{1}{\overline{G}}\left(\frac{\overline{G}}{G}\frac{\partial \overline{x}^p}{\partial x^j}\right)\frac{\partial}{\partial \overline{x}^p}\left(GF^j\right)$$

move parenthetical expression inside derivative

$$= \frac{1}{\overline{G}}\frac{\partial}{\partial \overline{x}^p}\left(\overline{G}\overline{F}^p\right) - F^j\left\{\frac{G}{\overline{G}}\frac{\partial}{\partial \overline{x}^p}\left(\frac{\overline{G}}{G}\frac{\partial \overline{x}^p}{\partial x^j}\right)\right\}$$

where

$$\overline{F}^p = \alpha\,\overline{g}^{pk}\left(\partial f / \partial \overline{x}^k\right)$$

$$\rightarrow \quad \nabla \bullet \mathbf{F} = \overline{\nabla} \bullet \overline{\mathbf{F}} - F^j\left\{\frac{G}{\overline{G}}\frac{\partial}{\partial \overline{x}^p}\left(\frac{\overline{G}}{G}\frac{\partial \overline{x}^p}{\partial x^j}\right)\right\}$$

components $j=0,1,2,3$

GCL: ↓

Since in general, the F^j are arbitrary, invariance of divergence →

$$\frac{G}{\overline{G}}\frac{\partial}{\partial \overline{x}^p}\left(\frac{\overline{G}}{G}\frac{\partial \overline{x}^p}{\partial x^j}\right) \equiv 0$$

GCL is required for conservation laws

IMPLEMENTATION of GA in EULAG

- Vertical mapping is analytically specified, extended terrain-following coordinates. Vertical boundaries may be computed, however (Wedi & Smolarkiewicz, *JCP* 2004; Ortiz & Smolarkiewicz, *IJNMF* 2006)

- Horizontal mappings from $\mathbf{S}_t$ to $\mathbf{S}_p$ may be specified analytically, computed numerically, or be a mix (hybrid).

- This separability of the horizontal vs. vertical mappings translates into the metric identities as well as the Jacobian

EULAG implementation, cont.

- HORIZONTAL GA:

Numerical transformations

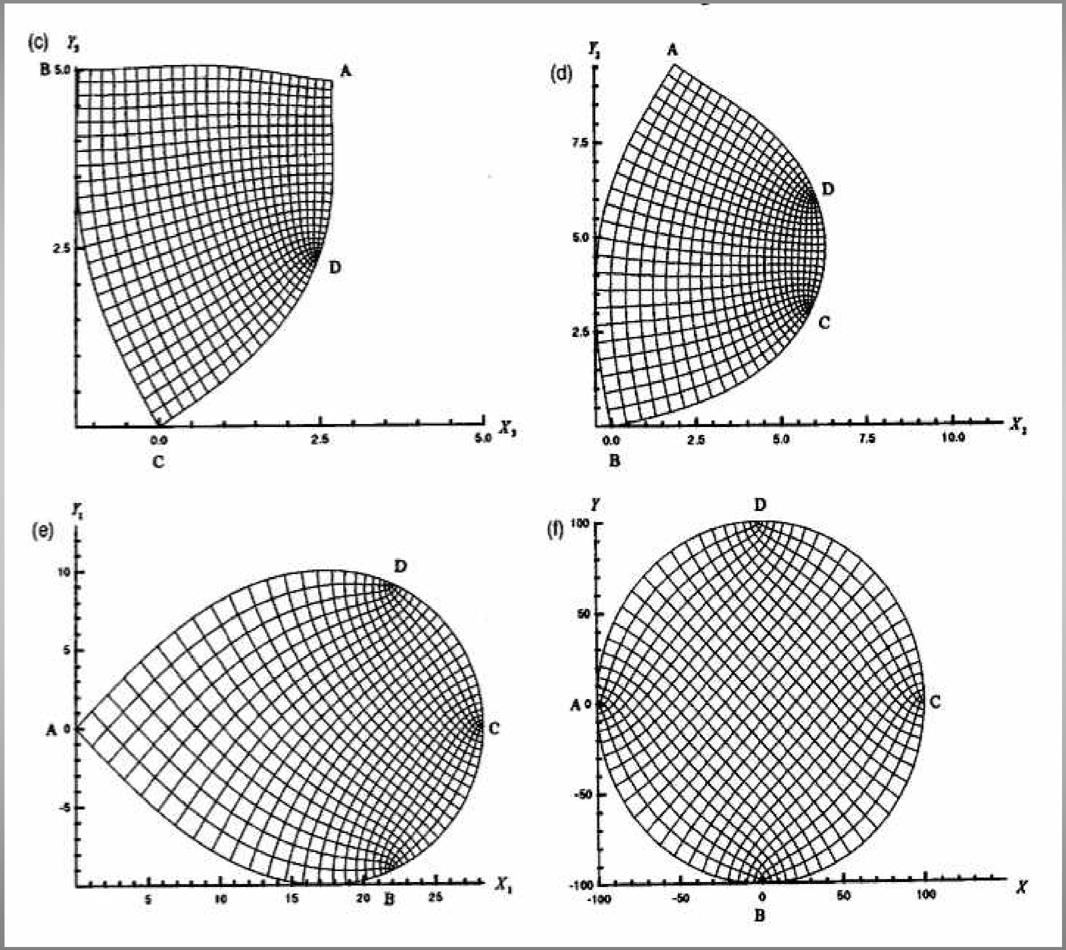
1. BODY FITTED COORDINATES

of Thompson et.al (*JCP* 1974), generated transformed coordinates $(\bar{x},\bar{y})$ via the numerical solution of coupled Poisson equations with Dirichlet boundary conditions, one for each coordinate:

$$\nabla^2 x = P(x,y), \quad \nabla^2 y = Q(x,y)$$

where *P* and *Q* are source functions used to control the grid interior

horizontal GA, cont.



“elliptic” generator example

(albeit solved via boundary element method; Tsay & Hsu, *IJNME* 1997)

horizontal GA, cont.

BASIC CONFLICT arises in GA:

- FLOW FEATURES can require very high resolution in isolated, distinct regions

$$\Delta x_{\max} / \Delta x_{\min} \rightarrow larger$$

- GRID QUALITY encompasses smoothness, orthogonality, monotonicity…(impact TE, Thompson & Mastin, *ASME* 1983; stability)

$$\Delta x_{\max} / \Delta x_{\min} \rightarrow unity$$

- COURANT NUMBER limitations will be set by smallest grid interval → adaption in time

horizontal GA, cont.

2. VARIATIONAL METHODS

used to develop elliptic grid generators (Brackbill and Saltzman *JCP* 1982)

Extremize: $I = \int_A f(\bar{x}, \bar{y}, x, x_{\bar{x}}, x_{\bar{y}}, \ldots)\, d\bar{x}\, d\bar{y} \rightarrow$

Euler-Lagrange Eq: $\frac{\partial f}{\partial x} - \frac{\partial}{\partial \bar{x}}\left(\frac{\partial f}{\partial x_{\bar{x}}}\right) - \frac{\partial}{\partial \bar{y}}\left(\frac{\partial f}{\partial x_{\bar{y}}}\right) + \ \ldots \ = 0$

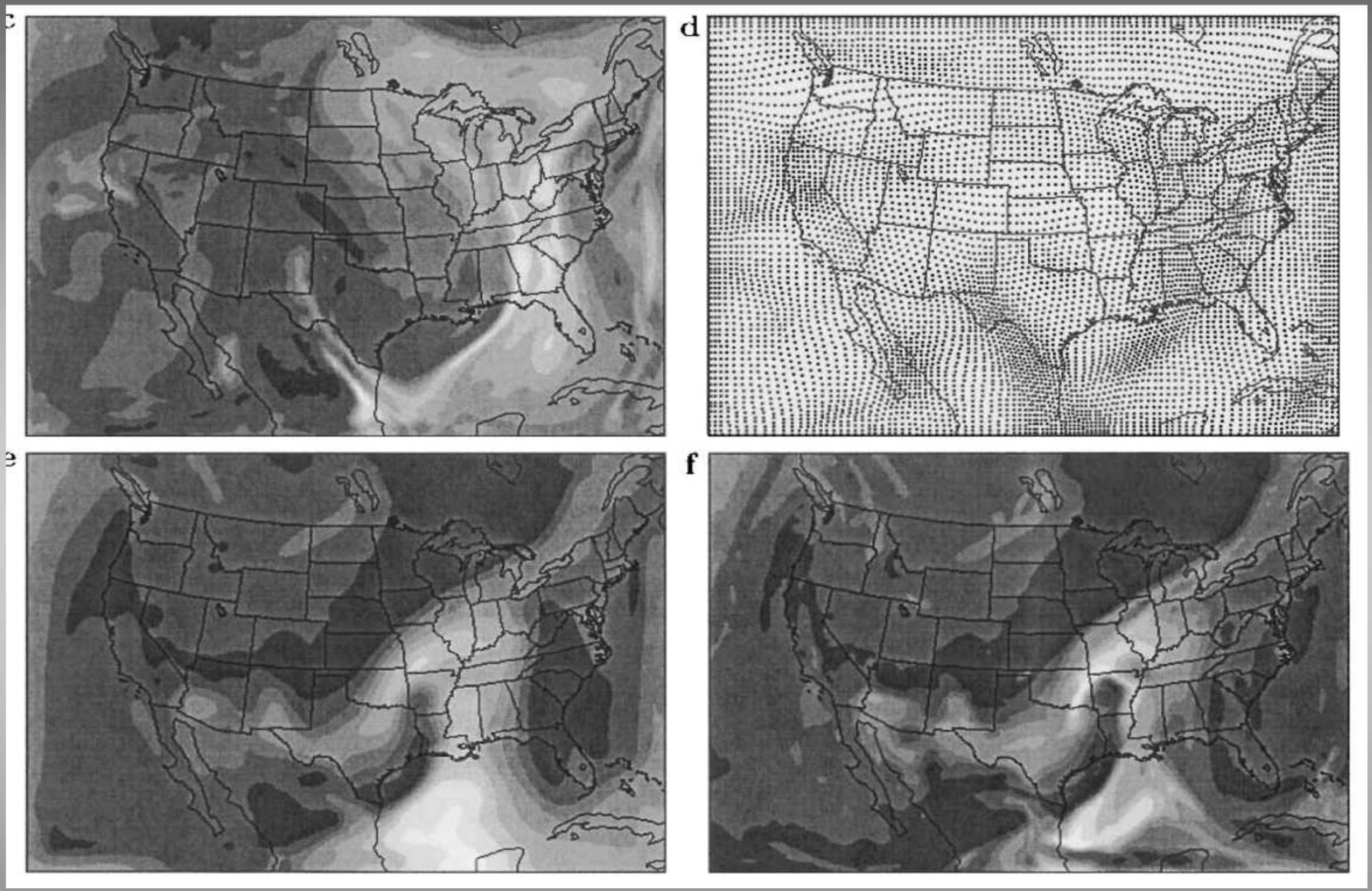
(Weinstock, *Dover P*. 1974)

1D example: $f = w(\bar{x}) \bullet (x_{\bar{x}})^2 / 2 \rightarrow$ $\frac{\partial}{\partial \bar{x}}\left(w \frac{\partial x}{\partial \bar{x}}\right) = 0$

weight function ↑

$$\rightarrow x(\bar{x}) = x_L + c \int_{\bar{x}_L}^{\bar{x}} w(\chi)^{-1} d\chi$$

EQUIDISTRIBUTION (Dietachmayer, *MWR* 1992)



Passive tracer advection at 2 and 5 days: (e) leapfrog MM5, (c,d,f) MPDATA with GA (Iselin et. al, *MWR* 2002,2005)

horizontal GA, cont.

3. NFT GRID GENERATION

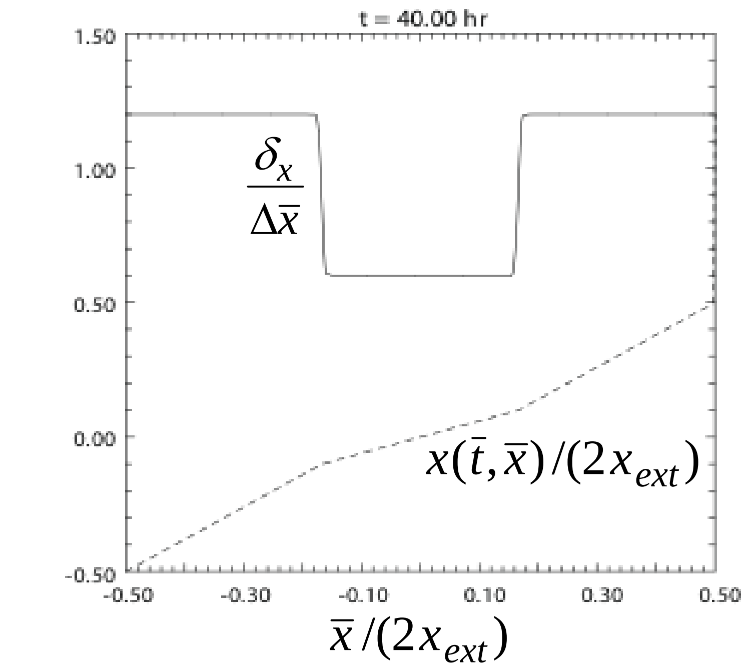
advection of *grid point density* via MPDATA

(Prusa & Smolarkiewicz, *JCP* 2003)

If $\partial U / \partial \bar{x} = 0$ then the NFT solution for mesh density,

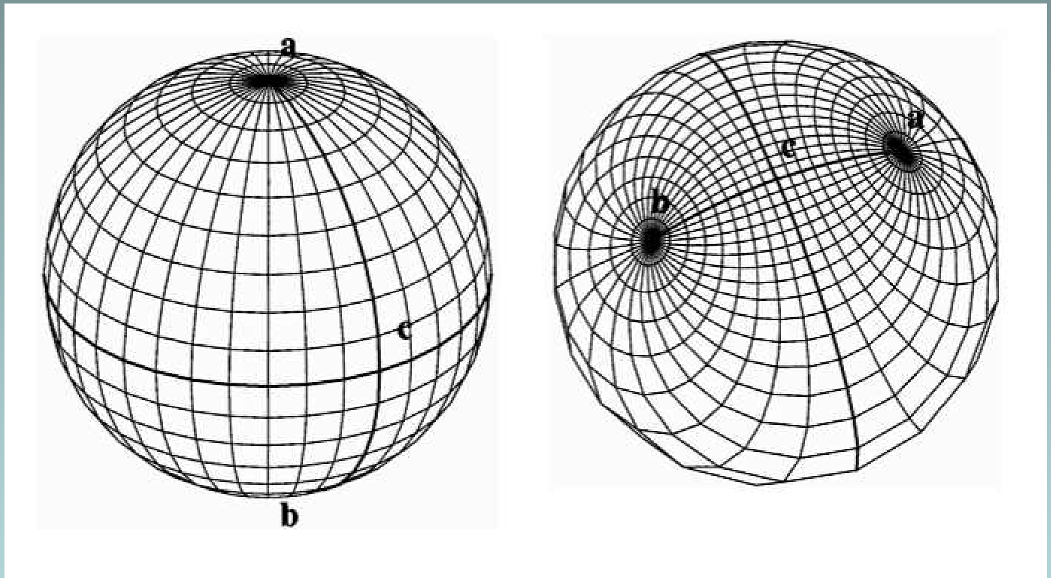
$$\frac{\partial \delta_x}{\partial \bar{t}} + \frac{\partial (U \delta_x)}{\partial \bar{x}} = 0$$

where $\delta_x \sim \partial x / \partial \bar{x}$ will be conservative and monotone



Analytical transformations

1. CONFORMAL MAPPING



Linear fractional transformation on the sphere (Bentsen et. al, *MWR* 1999)

Schwarz-Christoffel transformation: can transform a simple domain into an n-sided polygon (Case, *SIAM News* 2008)

horizontal GA, cont.

2. "ALGEBRAIC" MAPPINGS

not as flexible as fully numerical generation BUT …

- Core set of mappings is coded and ready to use
- Offer considerable speed advantage
- Easy to control grid properties by defining mappings with "tunable" parameters.

$$X\left(\overline{X} \mid S_x(\bar{t}), X_o(\bar{t})\right) = X_o + S_x^{-1}(\overline{X} - \overline{X}_o) + \frac{(1 - S_x^{-1})(\overline{X} - \overline{X}_o)^5}{(1 + 10\overline{X}_o^2 + 5\overline{X}_o^4)}$$

$$X_o = 1 - S_x^{-1}(1 - \overline{X}_o) - \frac{(1 - S_x^{-1})(1 - \overline{X}_o)^5}{(1 + 10\overline{X}_o^2 + 5\overline{X}_o^4)}$$

Mapping 2:

from *function xmap1*: open domain, 1D unimodal

horizontal GA, cont.

- extensions to 2D transformations

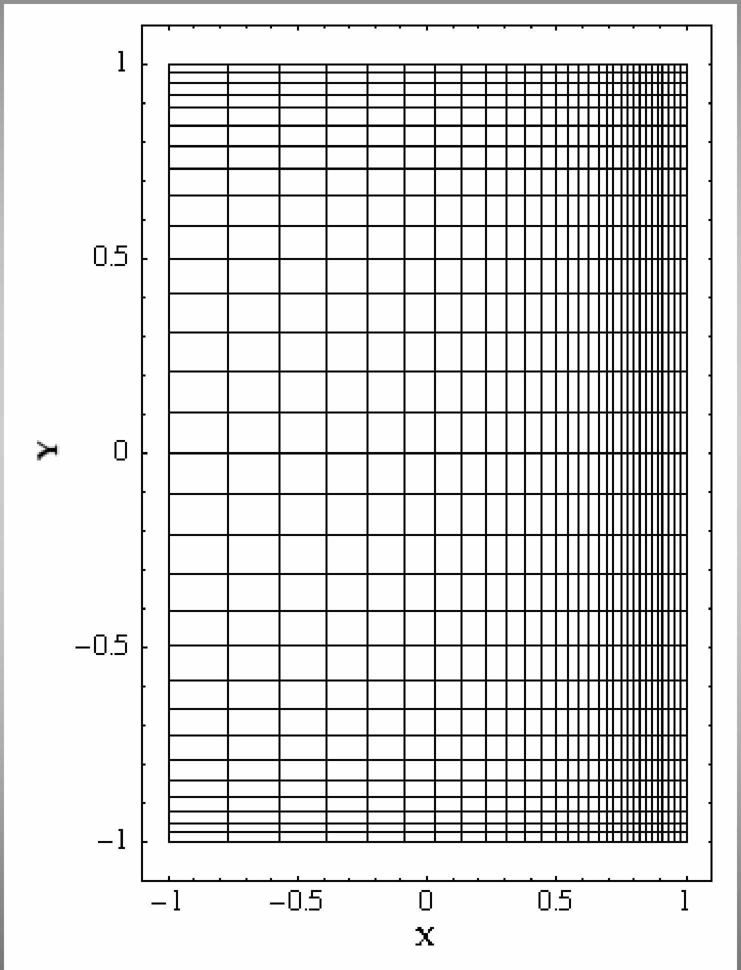
$$X'\left(\overline{X},\overline{Y} \mid S_x(\bar{t}),X_o(\bar{t})\right)= f_o(\overline{Y})\bullet X\left(\overline{X} \mid S_x(\bar{t}),X_o(\bar{t})\right)+ f_1(\overline{Y})\bullet \overline{X}$$

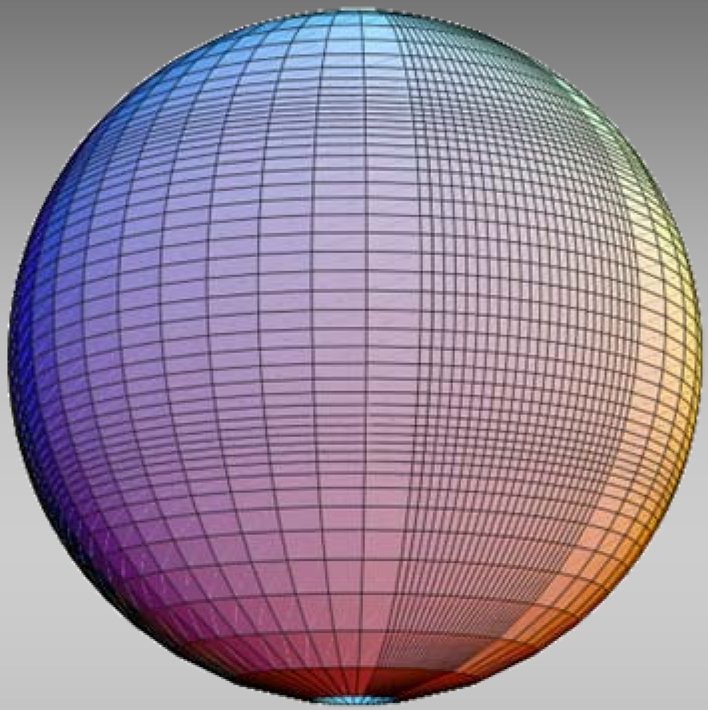
$$f_o(\overline{Y})=1-\overline{Y}^4(3-2\overline{Y}^2), \quad f_1(\overline{Y})=1- f_o(\overline{Y})$$

Mapping 5: from *function xmap1*: x-open, y-open domain, 2D unimodal

- Flatness properties ($\partial^p X/\partial\overline{X}^p = 0$) in region of enhanced resolution
- Parameters can be determined numerically for dynamic adaptivity.
- More sophisticated mappings can be built from a relatively small set of basic transformations

horizontal GA, cont.





- Map 3: *ymap1*: open 1D bimodal
- (left) Map 2: xmap1, open 1D unimodal
- (above) xmap2, periodic 1D nest

QuickTime™ and a
Animation decompressor
are needed to see this picture.

Map 5: *xmap1 & ymap1*: x-open, y-open domain, 2D unimodal

EULAG implementation, cont.

- KD IDENTITIES:

"hard wired" into code (*metryc*)

1. HORIZONTAL $\partial \bar{x}^m / \partial x^j$ (*m*=*1,2*)

$$E_{,x} = \mathrm{D}_{,\bar{y}} / \overline{G}_{xy}$$

$$D_{,x} = -\mathrm{D}_{,\bar{x}} / \overline{G}_{xy}$$

$$E_{,y} = -\mathrm{E}_{,\bar{y}} / \overline{G}_{xy}$$

$$D_{,y} = \mathrm{E}_{,\bar{x}} / \overline{G}_{xy}$$

required for

$$\tilde{G}_j^k := \sqrt{g^{jj}} \left(\partial \bar{x}^k / \partial x^j \right)$$

grid speeds

$$E_{,t} = \left(\mathrm{E}_{,\bar{y}} \mathrm{D}_{,\bar{t}} - \mathrm{D}_{,\bar{y}} \mathrm{E}_{,\bar{t}} \right) / \overline{G}_{xy}$$

$$D_{,t} = \left(\mathrm{D}_{,\bar{x}} \mathrm{E}_{,\bar{t}} - \mathrm{E}_{,\bar{x}} \mathrm{D}_{,\bar{t}} \right) / \overline{G}_{xy}$$

recall

$$\overline{G}_{xy} = \left(\mathrm{E}_{,\bar{x}} \mathrm{D}_{,\bar{y}} - \mathrm{D}_{,\bar{x}} \mathrm{E}_{,\bar{y}} \right)$$

KD identities, cont.

2. VERTICAL $\partial \bar{x}^m / \partial x^j$ (m=3)

Analytical expressions coded that satisfy identities exactly, but only for case G_{xy} = 1

e.g., $C_{,x} = -C_{,\bar{x}} / C_{,\bar{z}}$, $C_{,y} = -C_{,\bar{y}} / C_{,\bar{z}}$, in lieu of

$$C_{,x} = \frac{C_{,\bar{y}} D_{,\bar{x}} - C_{,\bar{x}} D_{,\bar{y}}}{C_{,\bar{z}} G_{xy}} , \quad C_{,y} = \frac{C_{,\bar{x}} E_{,\bar{y}} - C_{,\bar{y}} E_{,\bar{x}}}{C_{,\bar{z}} G_{xy}} \quad (j = 1,2)$$

grid speed is simularly cons. directly from $z_{s,t}$, $H_{s,t}$

component j=3 $C_{,z} = C_{,\bar{z}}^{-1}$ is perfect

EULAG implementation, cont.

- GCL IDENTITIES:

"under development" - satisfied perfectly in some cases, but not generally

$$\frac{G}{\overline{\overline{G}}}\frac{\partial}{\partial \bar{x}^p}\left(\frac{\overline{G}}{G}\frac{\partial \bar{x}^p}{\partial x^j}\right) \equiv 0 \quad \rightarrow \quad \frac{1}{\overline{G}'}\frac{\partial}{\partial \bar{x}^p}\left(\overline{G}'\frac{\partial \bar{x}^p}{\partial t}\right) \equiv 0$$

where $\overline{G}' =: \overline{G}_o \overline{G}_{xy}$

Consider the GCL component *j*=0:

$$\rightarrow \quad \frac{1}{\overline{G}'}\left\{\frac{\partial \overline{G}'}{\partial \bar{t}} + \frac{\partial(\overline{G}' E_{,t})}{\partial \bar{x}} + \frac{\partial(\overline{G}' D_{,t})}{\partial \bar{y}} + \frac{\partial(\overline{G}' C_{,t})}{\partial \bar{z}}\right\} = 0$$

Often considered "THE" GCL

(Thomas and Lombard, *AIAA* 1979)

GCL identities, cont.

ADDITIONAL PERSPECTIVES ...

- Generally, cannot ignore components j = 1,2,3 !
- Diagnostically, consider spatial terms a divergence $\rightarrow$ can use *subroutine rhsdiv*

$$\frac{1}{\overline{G}'}\frac{\partial \overline{G}'}{\partial \bar{t}}+\frac{1}{\overline{G}'}\left\{\frac{\partial(\overline{G}'E_{,t})}{\partial \bar{x}}+\frac{\partial(\overline{G}'D_{,t})}{\partial \bar{y}}+\frac{\partial(\overline{G}'C_{,t})}{\partial \bar{z}}\right\}=0$$

- Possible to consider GCL as “elliptic BVP” rather than prognostic

GCL identities, cont.

Consider the j = 3 component:

$$\frac{1}{\overline{G}'}\frac{\partial}{\partial \bar{x}^p}\left(\overline{G}'\frac{\partial \bar{x}^p}{\partial z}\right) \equiv 0$$

$$\rightarrow \frac{1}{\overline{G}'}\frac{\partial(\overline{G}'C_{,z})}{\partial \bar{z}} = \frac{1}{\overline{G}_o}\frac{\partial(\overline{G}_o C_{,z})}{\partial \bar{z}} = 0$$

recall $\bar{z} = C(\xi)$ where $\xi = H_o(z - z_s)/(H - z_s)$

$$\rightarrow \quad C_{,z} = \frac{dC}{d\xi}\frac{\partial \xi}{\partial z} = \frac{dC}{d\xi}\left(\frac{H_o}{H - z_s}\right) = \overline{G}_o^{-1}$$

→ In *EULAG*, the j = 3 component of the GCL is satisfied identically

GCL identities, cont.

VERTICAL TRANSFORMATIONS ONLY:

(case $\overline{G}_{xy} = 1$; j = 0,1,2)

$$\frac{1}{\overline{G}_o}\frac{\partial}{\partial \overline{x}^p}\left(\overline{G}_o \frac{\partial \overline{x}^p}{\partial x^j}\right) \equiv 0 \quad \rightarrow \quad \frac{\partial}{\partial \overline{x}^p}\left(\frac{\partial \overline{x}^p}{\partial x^j}\right) = -\frac{1}{\overline{G}_o}\frac{\partial \overline{G}_o}{\partial x^j}$$

$$\rightarrow \quad \frac{\partial}{\partial \overline{z}}\left(\frac{\partial \overline{z}}{\partial x^j}\right) = \frac{\partial C_{,x^j}}{\partial \overline{z}} = -\frac{1}{\overline{G}_o}\frac{\partial \overline{G}_o}{\partial x^j}$$

(Prusa & Gutowski, *IJNMF* 2006)

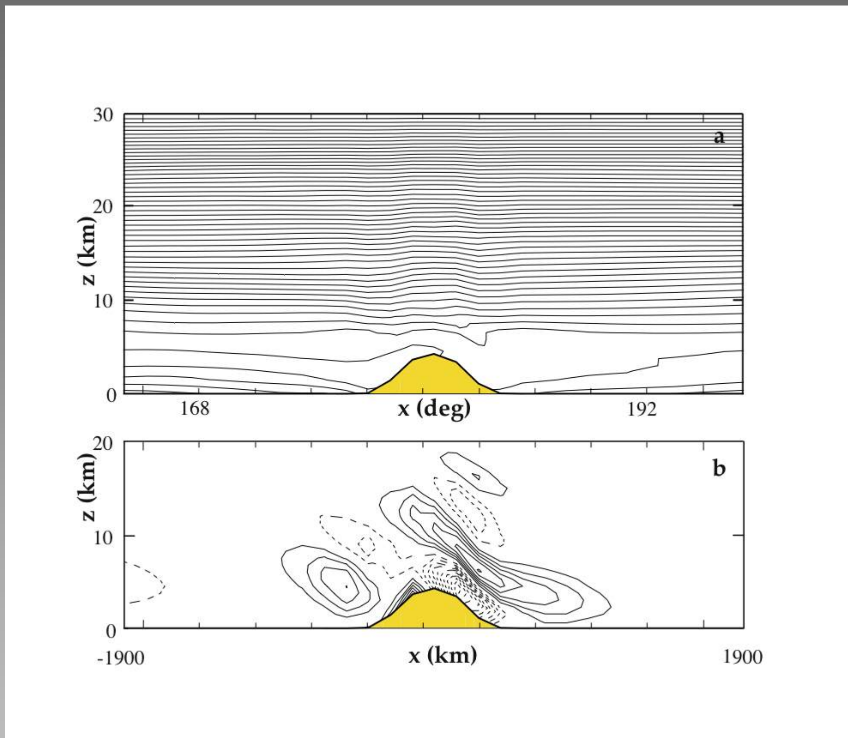
$$C_{,x^j} = -\frac{1}{\overline{G}_o}\left\{\frac{\partial z_s}{\partial x^j} + \xi\left(\frac{\partial H}{\partial x^j} - \frac{\partial z_s}{\partial x^j}\right)H_o^{-1}\right\}$$

$$\rightarrow \quad \frac{\partial C_{,x^j}}{\partial \overline{z}} = -\left(\frac{H_{,x^j} - z_{s,x^j}}{H - z_s}\right) = -\frac{1}{\overline{G}_o}\frac{\partial \overline{G}_o}{\partial x^j}$$

These expressions are employed directly in code for j = 1,2

Time derivatives, j = 0, handled case by case

vertical, cont.



Test case: flow over idealized mountain range (Prusa & Gutowski, *IJNMF* 2006)

S_x = 280 km,
S_y = 80 km, A~5 km

Domain is 3800 x 800 x 30 km

dx=dy=20 km,
dz=0.75 km (uniform)

	MAX	MIN	AVE	SD
GCL_t	0	0	0	0
GCL_x	0.1193e-15	-0.1265e-15	0.5919e-19	0.3147e-16
GCL_y	0.2590e-15	-0.3002e-15	-0.1155e-19	0.3147e-16

GCL identities, cont. (case $\overline{G}_o = 1$; j=1,2)

HORIZONTAL TRANSFORMATIONS

$$\frac{1}{\overline{G}_{xy}} \frac{\partial}{\partial \overline{x}^p}\left(\overline{G}_{xy} \frac{\partial \overline{x}^p}{\partial x^j}\right) \equiv 0 \quad \rightarrow \quad \frac{1}{\overline{G}_{xy}}\left\{\frac{\partial(\overline{G}_{xy} E_{,x^j})}{\partial \overline{x}} + \frac{\partial(\overline{G}_{xy} D_{,x^j})}{\partial \overline{y}}\right\} = 0$$

recall KD identities:

$$\overline{G}_{xy} E_{,x} = \mathrm{D}_{,\overline{y}} , \quad \overline{G}_{xy} D_{,x} = -\mathrm{D}_{,\overline{x}}$$

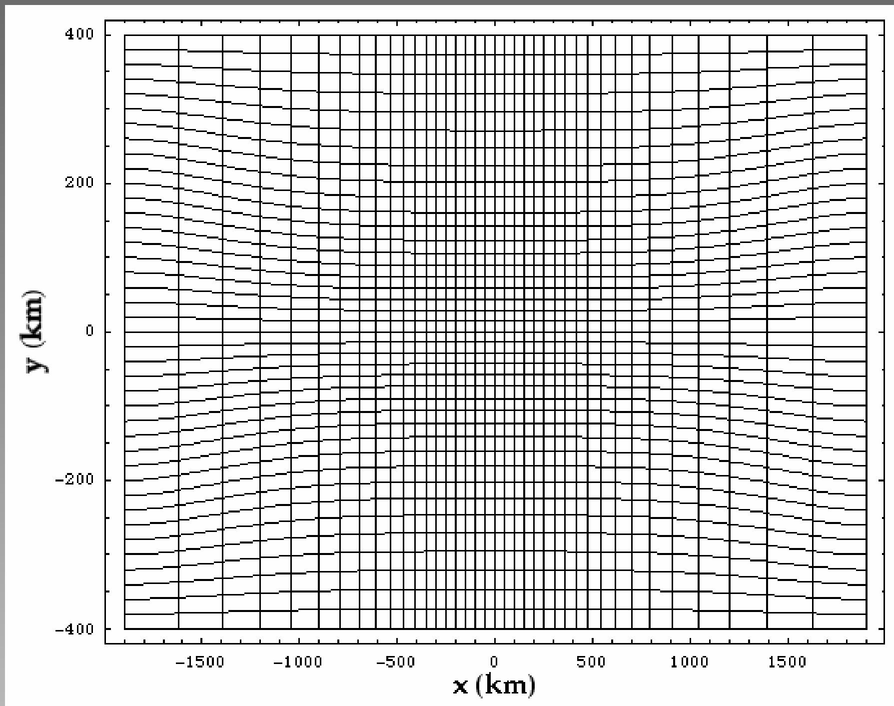
$$\overline{G}_{xy} E_{,y} = -\mathrm{E}_{,\overline{y}} , \quad \overline{G}_{xy} D_{,y} = \mathrm{E}_{,\overline{x}}$$

$$\rightarrow \quad \left(\frac{\partial \mathrm{D}_{,\overline{y}}}{\partial \overline{x}} - \frac{\partial \mathrm{D}_{,\overline{x}}}{\partial \overline{y}}\right) = 0 , \quad \left(-\frac{\partial \mathrm{E}_{,\overline{y}}}{\partial \overline{x}} + \frac{\partial \mathrm{E}_{,\overline{x}}}{\partial \overline{y}}\right) = 0$$

j=1 j=2

$\Rightarrow$ COMMUTATIVITY of partial derivatives!

horizontal, cont.



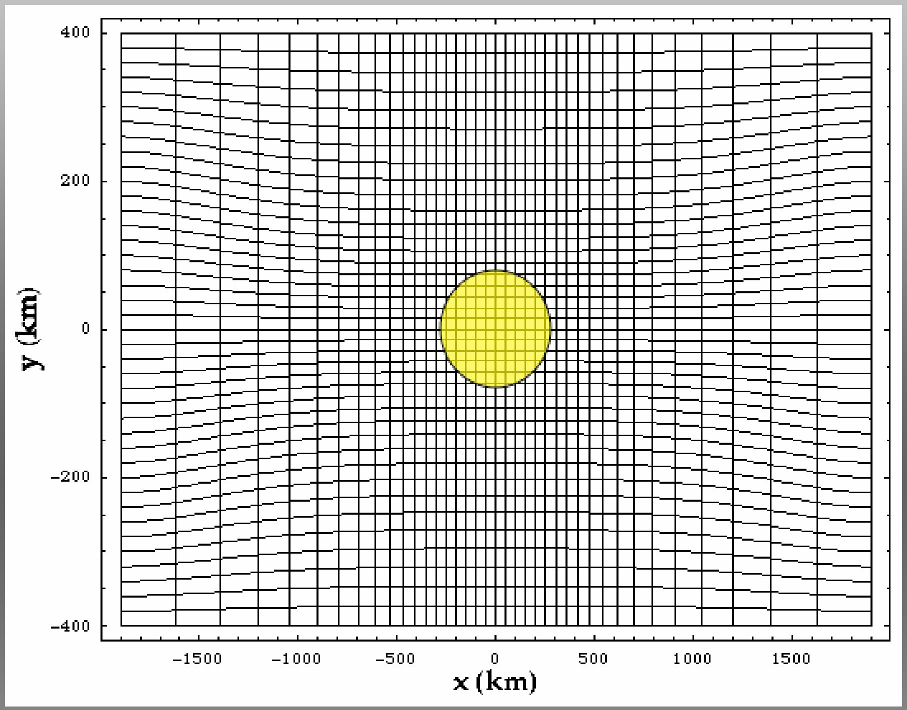
Test case: same flow as previously, only $A \to 0$, S_x=2, $S_y=2^{1/2}$;

x is 1D, unimodal
y is 2D, unimodal

Max $|v|$=0.2493e-08
Max $|w|$=0.6623e-10

	MAX	MIN	AVE	SD
GCL_t	0	0	0	0
GCL_x	0.6756e-15	-0.6756e-15	0.5237e-34	0.5521e-16
GCL_y	0.2891e-15	-0.2764e-15	0.3470e-19	0.5521e-16

Horizontal and vertical test of GCL



Test case: same initial flow, but:

topography, A ~ 5 km, l_x=280 km, l_y= 80 km

grid stretching, S_x=2, S_y=$2^{1/2}$, x is 1D, unimodal and y is 2D, unimodal

	MAX	MIN	AVE	SD
GCL_t	0	0	0	0
GCL_x	0.2452e-04	-0.2452e-04	-0.1591e-23	*
GCL_y	0.8880e-15	-0.9826e-15	-0.2476e-19	0.6682e-16

GCL identities, cont.

TIME ADAPTIVE TRANSFORMATIONS

Consider a case of 2D flow over topography, with a time adaptive, unimodal stretching function for x that concentrates resolution over the topography

$$\frac{1}{\overline{G}'}\frac{\partial}{\partial\overline{x}^p}\left(\overline{G}'\frac{\partial\overline{x}^p}{\partial x^j}\right)\equiv 0$$

Only $j = 0$ component is nontrivial; note $D_{,t}$, $C_{,t}=0$

$$\rightarrow \left\{\frac{1}{\overline{G}_o}\frac{\partial\overline{G}_o}{\partial\overline{t}}+\frac{1}{\overline{G}_{xy}}\frac{\partial\overline{G}_{xy}}{\partial\overline{t}}\right\}+\frac{1}{\overline{G}'}\frac{\partial\overline{G}'E_{,t}}{\partial\overline{x}}=0$$

$$\rightarrow \frac{\partial \mathrm{E}_{,\overline{x}}}{\partial\overline{t}}=-\frac{\mathrm{E}_{,\overline{x}}}{\overline{G}'}\frac{\partial(\overline{G}'E_{,t})}{\partial\overline{x}}-\frac{1}{\overline{G}_o}\frac{\partial\overline{G}_o}{\partial\overline{t}}$$

since $\overline{G}_{xy}=\mathrm{E}_{,\overline{x}}$

$$\rightarrow \frac{\partial^2\mathrm{E}_{,\overline{t}}}{\partial\overline{x}^2}=-\frac{\partial}{\partial\overline{x}}\left\{\frac{\mathrm{E}_{,\overline{x}}}{\overline{G}'}\frac{\partial(\overline{G}'E_{,t})}{\partial\overline{x}}\right\}+f(\overline{G}_o)$$

time GCL, cont.

Test case: same flow and mtn;

$S_x \rightarrow 1$ at t=0 to
2 at t=12 hr

x is 1D, unimodal

QuickTime™ and a
Animation decompressor
are needed to see this picture.

Ave flow changes
u_{ext} = 1.3%
w_{ext} = 2.6%

Note mountain *does not change*!

$GCL_{t,0}$ is uncorrected; $GCL_{t,I}$ is with elliptic iterations, but $f(G_o)$=0

	MAX	MIN	AVE	SD
$GCL_{t,0}$	0.4391e-03	-0.1099e-16	0.5738e-04	0.1217e-03
$GCL_{t,I}$	0.1008e-04	-0.1008e-04	0.3144e-08	0.8305e-05
GCL_x	0.1215e-15	-0.1106e-15	-0.3588e-18	*

GCL identities, concluded

REMARKS on GA

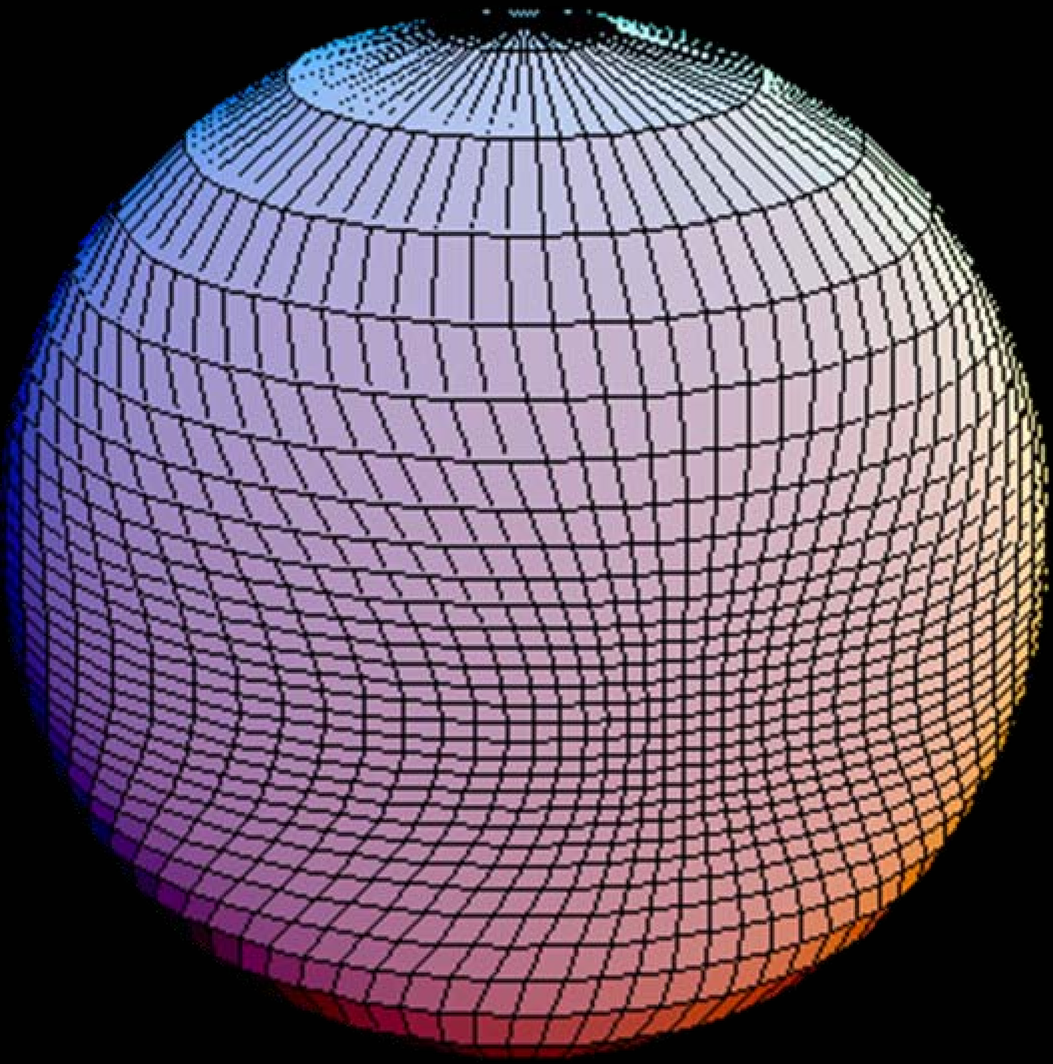
- Analytical mappings, when useful, have significant advantages

- KD identities can be easily solved, guarantee that tensors not including divergence transform properly

- GCL identities needed for divergence operator. Connected to commutativity.

Is solving a BVP better than integration of first order equations (i.e., a prognostic eq. for j=0)?

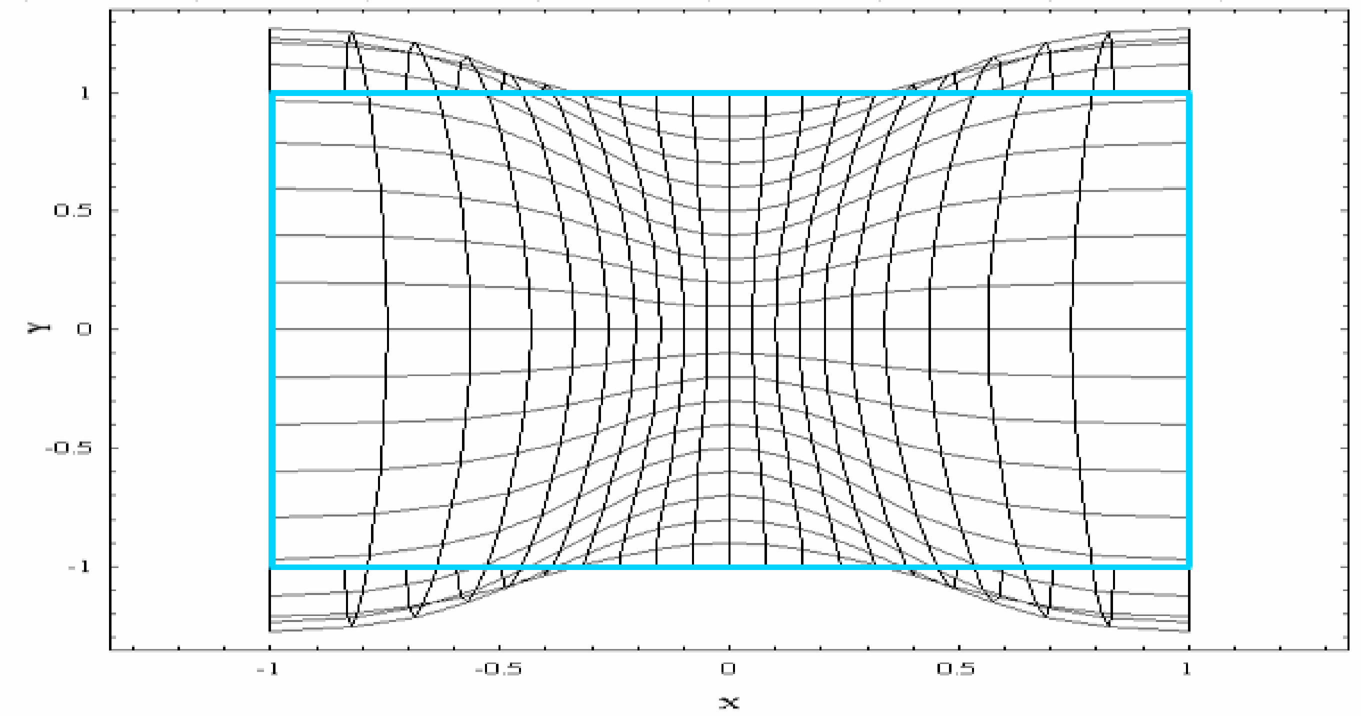
- Adaptation in time coordinate?



horizontal GA, cont.

Nonmonotonicity = DISASTER!

(inverse mapping no longer unique)



GA in Computational Models, cont.

- Consider an example from heat transfer

QuickTime™ and a
Animation decompressor
are needed to see this picture.

An isothermal block is suddenly heated at its surface:

1. From a global perspective, *coordinate x*, the surface heat transfer rate is initially unbounded

$$q = -k\, dT / dx$$

2. From a local perspective, with rescaled $X \sim x / t^{1/2}$, the surface temperature gradient is well behaved for all time.

$$(dT / dX)_{X=0} \sim -1/\sqrt{\pi}$$

→ define space so that “motion” looks simple

EULAG metrics, cont.

Separability of metric structure into given vertical and horizontal dependencies limits generality but offers:

- Built in analytical structure for vertical mapping helps maintain accuracy of vertical metric identities to machine precision
- All metric coefficients (except for Γ) can be computed from the product of one and two-dimensional arrays
- Separability can be used to help satisfy horizontal metric identities more easily