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A Godunov-type projection scheme for sound-proof models

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Structure

Numerics

Asymptotics

Conclusions

Structure

Compressible flow equations:

$$\boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \pi = -\rho g \boldsymbol{k}$$

$$\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) = 0$$

$$P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R$$

Structure

Pseudo-incompressible model: *(Durran (1988))*

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \overline{P} \nabla \pi = -\rho g \boldsymbol{k}$$

$$\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) = 0$$

$$\underline{P \equiv \overline{P}(z)}\,, \qquad \rho\theta = \overline{P}(z)\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

Structure

Compressible flow equations:

$$\boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \nabla p = -\rho g \boldsymbol{k}$$

$$\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) = 0$$

$$P = p^{\frac{1}{\gamma}} = \rho \theta$$

Structure

Anelastic model: *(Bannon (1996), from Dutton-Fichtl & Lipps-Hemler)*

$$\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$(\overline{\rho} \boldsymbol{v})_t + \nabla \cdot (\overline{\rho} \boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho} \nabla \pi = \frac{\theta'}{\overline{\theta}} \overline{\rho} g \boldsymbol{k}$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0$$

$$\underline{\rho \equiv \overline{\rho}(z)}\,, \qquad \overline{\rho}(z)\theta = P\,, \qquad \theta = \overline{\theta}(z) + \theta'\,, \qquad \pi = p'/\overline{\rho}(z)$$

Structure

Anelastic

$$
\begin{aligned}
\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) &= 0 \\
(\overline{\rho} \boldsymbol{v})_t + \nabla \cdot (\overline{\rho} \boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho} \nabla \pi &= \frac{\theta'}{\overline{\theta}} \overline{\rho} g \boldsymbol{k} \\
P_t + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

$$
\overline{\rho}(z)\theta = P \,, \qquad \theta = \overline{\theta}(z) + \theta'
$$

$$
\begin{aligned}
\nabla \cdot (\overline{\rho} \boldsymbol{v}) &= 0 \\
\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \pi &= \frac{\theta'}{\overline{\theta}} g \boldsymbol{k} \\
\theta_t + \boldsymbol{v} \cdot \nabla \theta &= 0
\end{aligned}
$$

$$
\theta' = \theta(z) - \overline{\theta}(z)
$$

Pseudo-incompressible

baroclinic torque / modified divergence

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \overline{P} \nabla \pi &= \frac{\theta'}{\overline{\theta}} \rho g \boldsymbol{k} \\
\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) &= 0
\end{aligned}
$$

$$
\rho(z)\theta = \overline{P} \,, \qquad \theta = \overline{\theta}(z) + \theta'
$$

$$
\begin{aligned}
(1/\theta)_t + \boldsymbol{v} \cdot \nabla (1/\theta) &= 0 \\
\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \underline{\theta \nabla \pi} &= \frac{\theta'}{\overline{\theta}} g \boldsymbol{k} \\
\underline{\nabla \cdot (\overline{P} \boldsymbol{v})} &= 0
\end{aligned}
$$

relevant for deep atmospheres / large scales*

*see, e.g., Smolarkiewicz & Dörnbrack (2007)

Structure

Anelastic

Boussinesq approximation

$$\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$(\overline{\rho} \boldsymbol{v})_t + \nabla \cdot (\overline{\rho} \boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho} \nabla \pi = \frac{\theta'}{\overline{\theta}} \overline{\rho} g \boldsymbol{k}$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0$$

$$\overline{\rho}(z)\theta = P\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \pi = \theta\, g \boldsymbol{k}$$

$$\theta_t + \boldsymbol{v} \cdot \nabla \theta = 0$$

Pseudo-incompressible

zero-Mach, variable density flow

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \overline{P} \nabla \pi = \frac{\theta'}{\overline{\theta}} \rho g \boldsymbol{k}$$

$$\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) = 0$$

$$\rho(z)\theta = \overline{P}\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\rho_t + \boldsymbol{v} \cdot \nabla \rho = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho} \nabla \pi = (\rho - \overline{\rho})\, g \boldsymbol{k}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

Small scale limits

Structure

Remarks:

- Sound-proof, **ad-hoc(?)** models involving **advection** and **internal waves**

- Bannon (1996) *(merging Dutton-Fichtl / Lipps-Hemler)*

  anelastic: $\rho \equiv \overline{\rho}(z) \quad \Rightarrow \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$

  small-scale limit: Boussinesq

- Durran (1988)

  pseudo-incompressible: $p^{\frac{1}{\gamma}} \equiv \overline{P}(z) \quad \Rightarrow \quad \nabla \cdot (\overline{\rho\theta} \boldsymbol{v}) = 0$

  small-scale limit: zero-Mach variable density

- Very close relatives of the full compressible flow equations

(?) see "Asymptotics" section

Structure

Numerics

Asymptotics

Conclusions

Numerics

Compressible flow equations:

$$\begin{aligned} \boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\ (\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \pi &= -\rho g \boldsymbol{k} \\ \boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) &= 0 \end{aligned}$$

$$P = p^{\frac{1}{\gamma}} = \rho \theta \,, \qquad \pi = p / \Gamma P \,, \qquad \Gamma = c_p / R$$

Numerics

Pseudo-incompressible model: ***(Durran (1988))***

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \overline{P} \nabla \boldsymbol{\pi} &= -\rho g \boldsymbol{k} \\
\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) &= 0
\end{aligned}
$$

$$
\underline{P \equiv \overline{P}(z)}\,, \qquad \rho\theta = \overline{P}(z)\,, \qquad \theta = \overline{\theta}(z) + \theta'
$$

Numerics

Auxiliary variable projection scheme* – predictor step

Solve over Δt by second-order (upwind FV) scheme

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) \;=\; 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \boldsymbol{q} \;=\; -\rho g \boldsymbol{k}$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) \;=\; 0$$

- $\boldsymbol{q} = \boldsymbol{\pi}^{\boldsymbol{n}}$:
  2nd-order advection of $\theta = P/\rho$ and $\boldsymbol{v} = (\rho \boldsymbol{v})/\rho$
- 2nd-order gravity term

* incompr. case, 2nd-ord.: Schneider et al., JCP '99; 4th-ord.: Kadioglu, Minion, Klein, JCP '08

Numerics

Auxiliary variable projection scheme – predictor step

Solve over Δt by second-order (upwind FV) scheme:

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \boldsymbol{q} &= -\rho g \boldsymbol{k} \\
P_t + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

- $\boldsymbol{q} = \boldsymbol{\pi}^{\boldsymbol{n}}$:
  2nd-order advection of $\theta = P/\rho$ and $\boldsymbol{v} = (\rho \boldsymbol{v})/\rho$ **for arbitrary $\nabla \cdot (P\boldsymbol{v})$!**
- 2nd-order gravity term

Numerics

Advection version 1:

solve

$$(\rho\theta)_t + \nabla \cdot (\rho\theta \boldsymbol{v}) = 0$$

making explicit use of

$$\rho \equiv \overline{\rho}(z) \qquad \text{and} \qquad \nabla \cdot (\overline{\rho} \boldsymbol{v}) \equiv 0$$

in 1D:

$$\theta_i^{n+1} = \frac{(\overline{\rho}\theta)_i^{n+1}}{\overline{\rho}} = \frac{1}{\overline{\rho}} \left\{ (\overline{\rho}\theta)_i^n - \frac{\Delta t}{\Delta x} \left((\overline{\rho}\theta u)_{i+\frac{1}{2}} - (\overline{\rho}\theta u)_{i-\frac{1}{2}} \right) \right\}$$

$$= \theta_i^n - \frac{\Delta t}{\Delta x} \left\{ \underline{\overline{u}_i \left(\theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}} \right)} + \underline{\frac{\overline{\boldsymbol{\theta}}_{\boldsymbol{i}}}{\overline{\rho}} \left(\rho u_{i+\frac{1}{2}} - \rho u_{i-\frac{1}{2}} \right)} \right\}$$

where

$$\overline{u}_i = \frac{1}{2\overline{\rho}} \left((\overline{\rho} u)_{i+\frac{1}{2}} + (\overline{\rho} u)_{i-\frac{1}{2}} \right) \qquad \text{and} \qquad \overline{\theta}_i = \frac{1}{2} \left(\theta_{i+\frac{1}{2}} + \theta_{i-\frac{1}{2}} \right)$$

Numerics

Advection version 2:

solve simultaneously

$$\begin{aligned}(\rho\theta)_t + \nabla\cdot(\rho\theta\boldsymbol{v}) &= 0\\ \rho_t + \nabla\cdot(\rho\boldsymbol{v}) &= 0\end{aligned}$$

making **no** use of

$$\rho \equiv \overline{\rho}(z) \qquad \text{or} \qquad \nabla\cdot(\overline{\rho}\boldsymbol{v}) \equiv 0$$

in 1D:

$$\theta_i^{n+1} = \frac{(\overline{\rho}\theta)_i^{n+1}}{\rho_i^{n+1}} = \frac{(\overline{\rho}\theta)_i^n - \frac{\Delta t}{\Delta x}\left((\overline{\rho}\theta u)_{i+\frac{1}{2}} - (\overline{\rho}\theta u)_{i-\frac{1}{2}}\right)}{\overline{\rho}_i^n - \frac{\Delta t}{\Delta x}\left((\overline{\rho}u)_{i+\frac{1}{2}} - (\overline{\rho}u)_{i-\frac{1}{2}}\right)}$$

$$= \theta_i^n - \frac{\Delta t}{\Delta x}\left\{\underline{\overline{u}_i\left(\theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}}\right)} + \underline{\frac{\boldsymbol{\overline{\theta}_i - \theta_i^n}}{\rho_i^n}\left(\rho u_{i+\frac{1}{2}} - \rho u_{i-\frac{1}{2}}\right)}\right\} + \text{h.o.t.}$$

Numerics

Advection version 2:

solve simultaneously

$$(\rho\theta)_t + \nabla\cdot(\rho\theta\boldsymbol{v}) \;=\; 0$$
$$\rho_t + \nabla\cdot(\rho\boldsymbol{v}) \;=\; 0$$

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$$\rho \equiv \overline{\rho}(z) \qquad \text{or} \qquad \nabla\cdot(\overline{\rho}\boldsymbol{v}) \equiv 0$$

in 1D:

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$$= \theta_i^n - \frac{\Delta t}{\Delta x}\left\{\underline{\overline{u}_i\Big(\theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}}\Big)} + \underline{\frac{\boldsymbol{\overline{\theta}_i - \theta_i^n}}{\rho_i^n}\Big(\rho u_{i+\frac{1}{2}} - \rho u_{i-\frac{1}{2}}\Big)}\right\} + \text{h.o.t.}$$

Second-order advection achieved for arbitrary divergence

Numerics

Auxiliary variable projection scheme – predictor step

Solve over Δt by second-order (upwind FV) scheme:

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

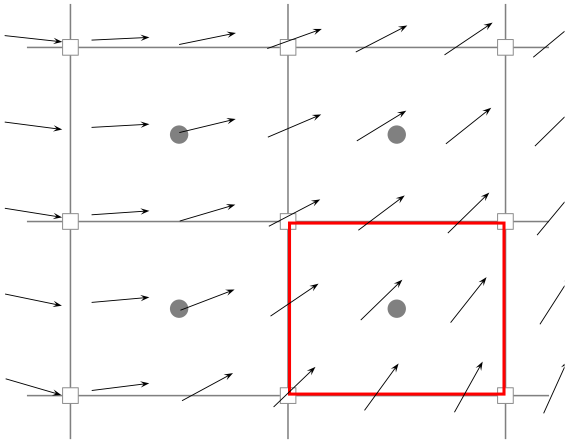
$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \boldsymbol{q} = -\rho g \boldsymbol{k}$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0$$

- $\boldsymbol{q} = \boldsymbol{\pi}^{\boldsymbol{n}}$:
  2nd-order advection of $\theta = P/\rho$ and $\boldsymbol{v} = (\rho \boldsymbol{v})/\rho$ **for arbitrary $\nabla \cdot (P v)$!**
- 2nd-order gravity term
- **first-order pressure gradient effect**
- $\boldsymbol{P^* - P^n = O((\Delta t)^2)}$

Numerics

Auxiliary variable projection scheme – MAC-projection



Constraint

$$\underline{\nabla \cdot (\overline{P}\boldsymbol{v})^{n+\frac{1}{2}} = 0} \quad \Rightarrow \quad \boldsymbol{\delta\pi} = \pi^{n+\frac{1}{2}} - \pi^{n}$$

Advection velocity correction (at cell interfaces)

$$(\overline{P}\boldsymbol{v})^{n+\frac{1}{2}} = (P\boldsymbol{v})^{*} - \frac{\Delta t}{2} P^{*}\theta^{*}\nabla\boldsymbol{\delta\pi}$$

Divergence control

$$\frac{1}{P^{n}}\nabla \cdot \left(P^{*}\theta^{*}\nabla\boldsymbol{\delta\pi}\right) = \frac{2}{(\Delta t)^{2}}\frac{1}{P^{n}}\left(P^{*} - P^{n}\right)$$

Post-correction of all advective flux contributions

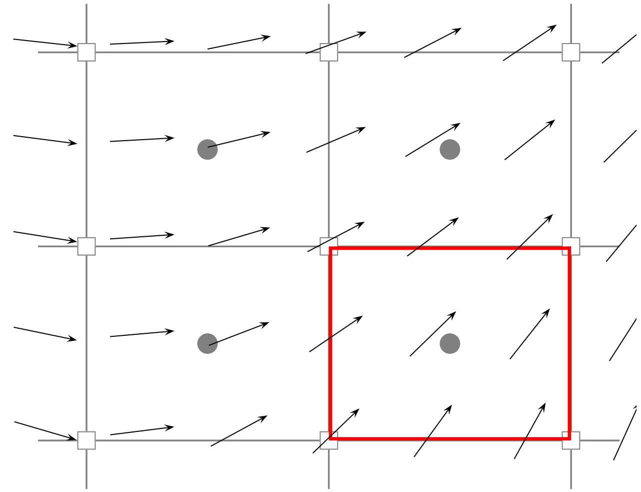
Numerics

Auxiliary variable projection scheme – MAC-projection

MAC-projection controls
advection fluxes across grid cell interfaces

Advection velocities result from some
interpolation from the cell-centers

Danger: **pressure-velocity decoupling**



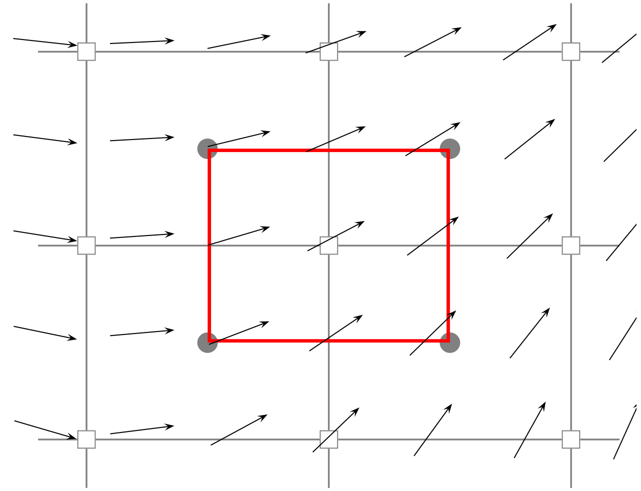
⇓

Previous approaches:

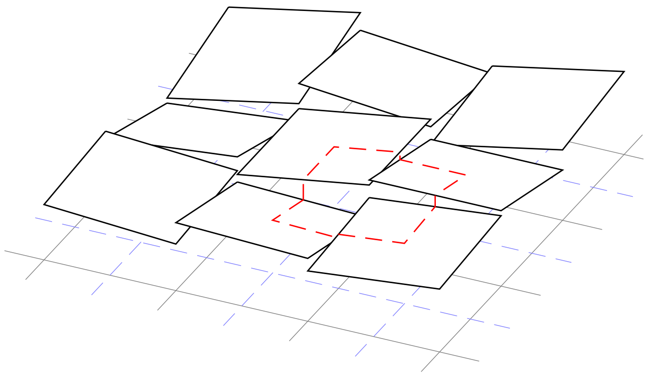
- Rhie-Chow stabilization — compromises mass continuity
- Exact projection — non-compact stencils
- Approximate projection — inexact divergence control

Numerics

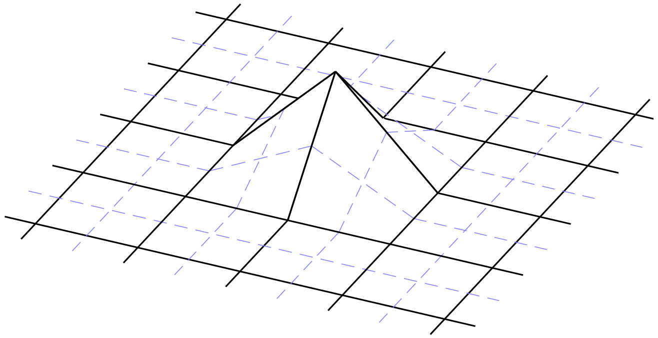
Alternative route*: cell-centered exact projection



Exact control of divergence on dual cells



Piecewise linear, discontinuous ansatz functions for momentum on primary cells



Bi-/trilinear pressure ansatz functions

* S. Vater, R.K., Num. Math., revised (2008)

Numerics

Stabilizing "Second projection":

inf-sup-stable Petrov-Galerkin scheme for generalized saddle-point problem

Find $(u, p) \in (\mathcal{X}_2 \times \mathcal{M}_1)$, such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b_1(\boldsymbol{v}, p) &= \langle f, v \rangle \qquad \forall\, \boldsymbol{v} \in \mathcal{X}_1 \\ b_2(\boldsymbol{u}, q) &= \langle g, q \rangle \qquad \forall\, q \in \mathcal{M}_2 \end{aligned}$$

where

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= (\boldsymbol{u}, \boldsymbol{v}) \\ b_1(\boldsymbol{v}, p) &= (\boldsymbol{v}, \nabla p) \\ b_2(\boldsymbol{u}, q) &= (\nabla \cdot \boldsymbol{u}, q) \end{aligned}$$

(result currently available for 2D shallow water / homentropic gasdynamics)

S. Vater, R.K., Num. Math., submitted (2007); using Nicolaïdes (1982), Bernardi et al. (1988)

Numerics

Summary of auxiliary-variable projection method

- **Predictor**
  - auxiliary **unconstrained** system
  - second-order upwind finite volume scheme
  - well-balanced discretization of gravity term
- **MAC-projection**
  - controls advective flux divergence
  - brings P back to $\overline{P}(z)$
- **cell-centered projection**
  - stabilizes checkerboard $p - \boldsymbol{v}$ modes
  - generates second-order accuracy w.r.t. pressure

Numerics

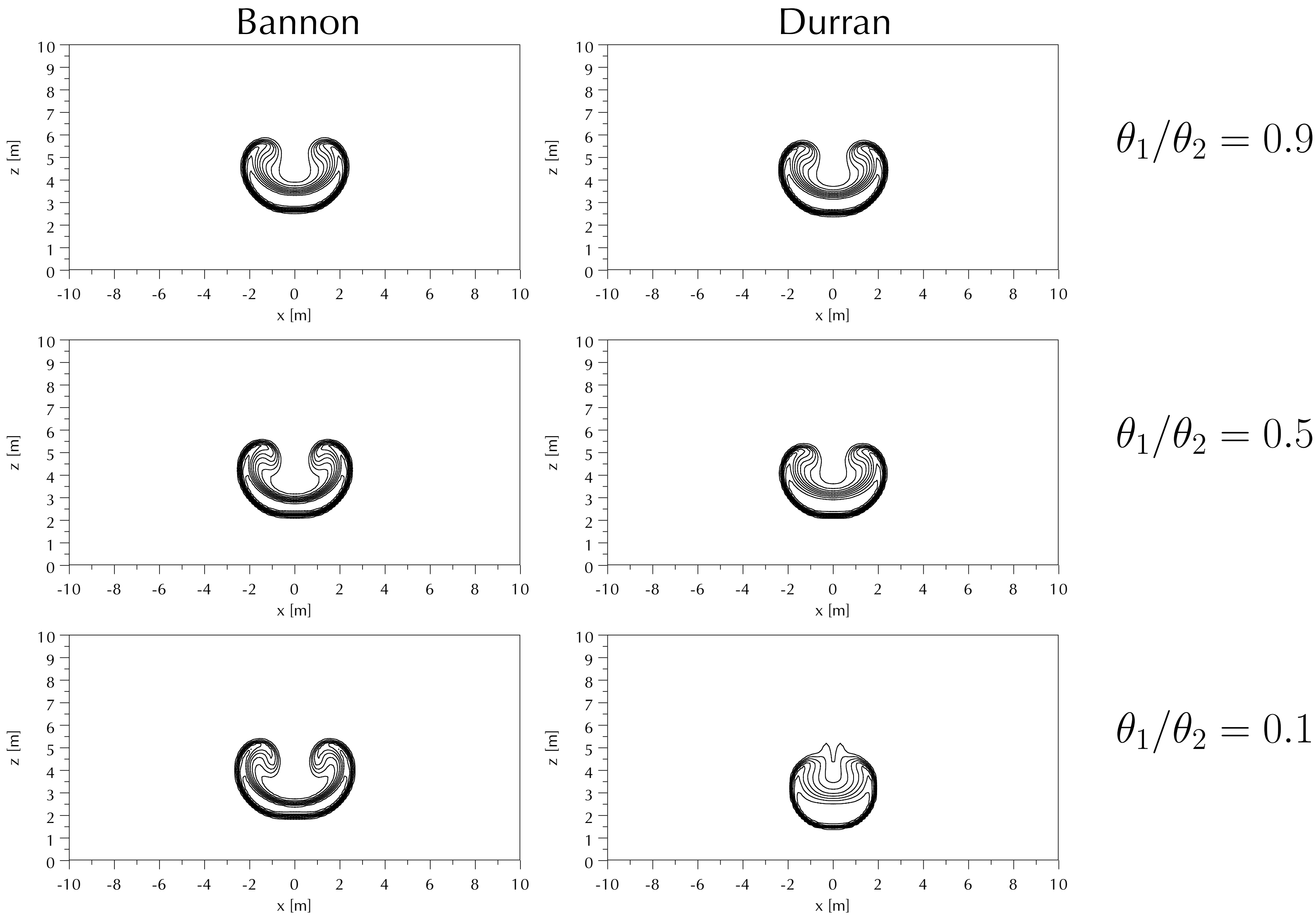
Summary of auxiliary-variable projection method

- **Predictor**
  - auxiliary **unconstrained** system
  - second-order upwind finite volume scheme
  - well-balanced discretization of gravity term
- **MAC-projection**
  - controls advective flux divergence
  - brings P back to $\overline{P}$
- **cell-centered projection**
  - stabilizes checkerboard $p - \boldsymbol{v}$ modes
  - generates second-order accuracy w.r.t. pressure

- **Background nowhere invoked explicitly!!**

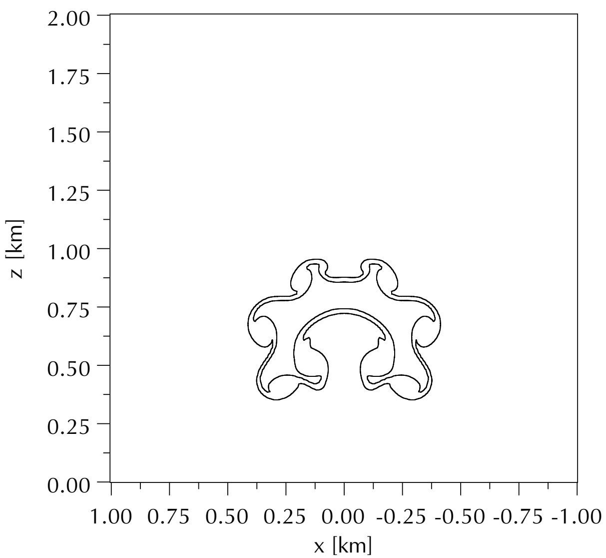
Numerics

Cold air blobs at small scales

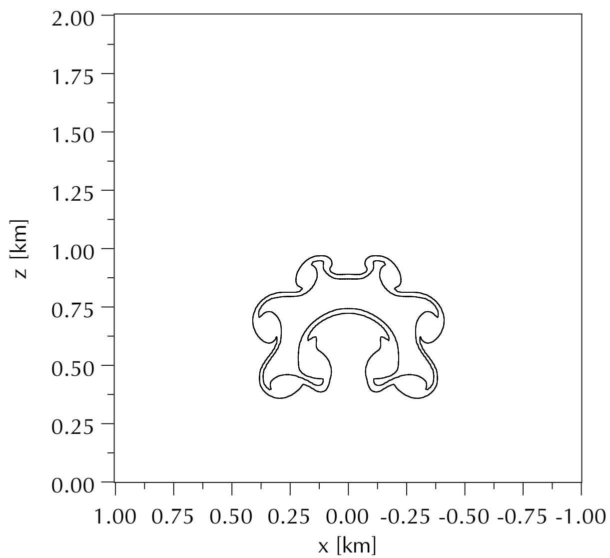


Numerics

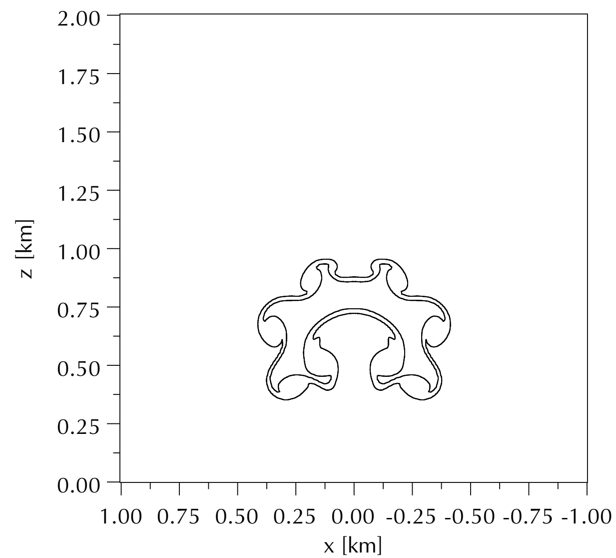
Rising sharp bubble, $\Delta\theta = 0.5\mathrm{K}$



Durran, $\mathrm{divtol} = 10^{-6}$

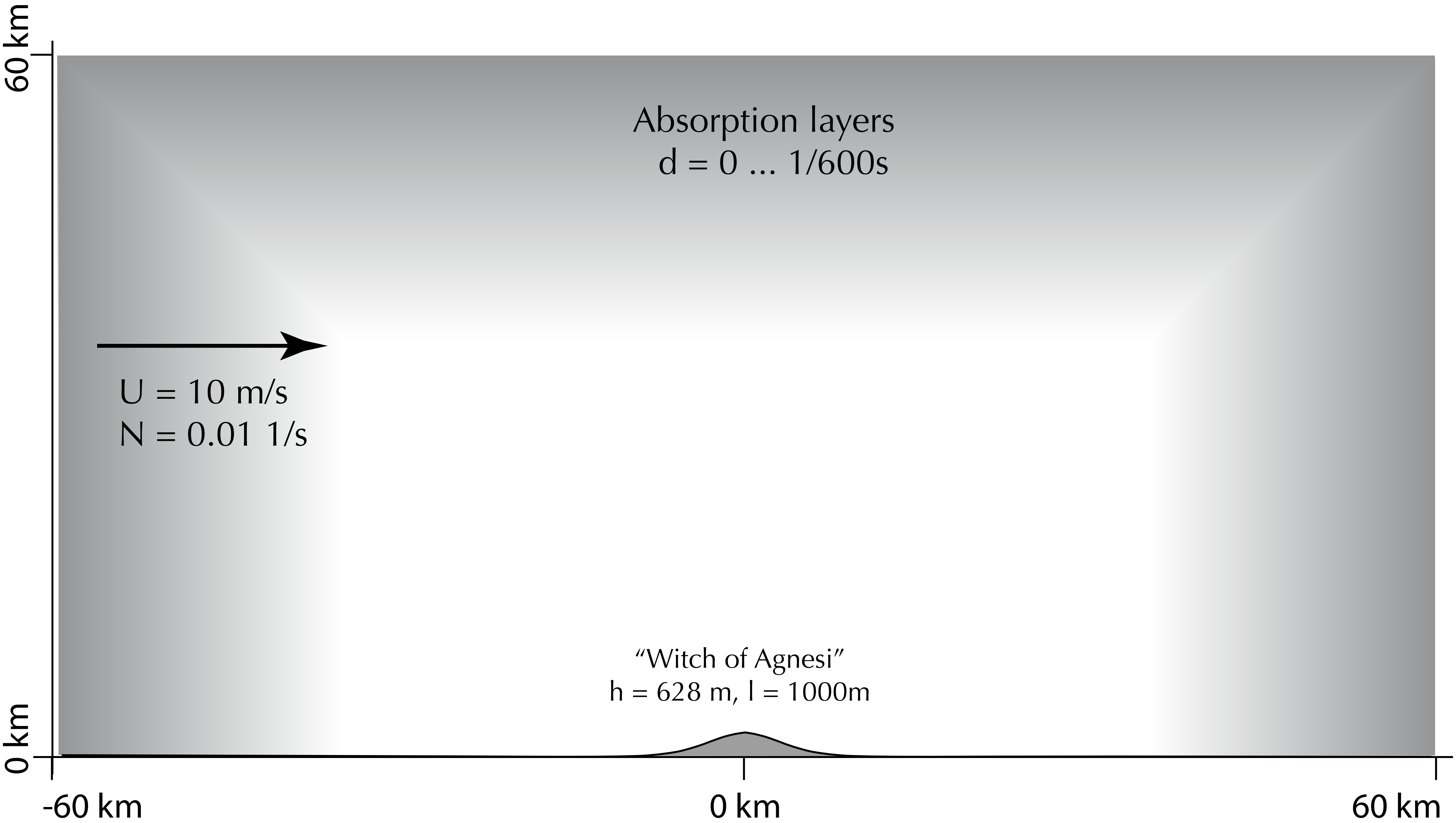


Durran, $\mathrm{divtol} = \mathbf{10^{-2}}$

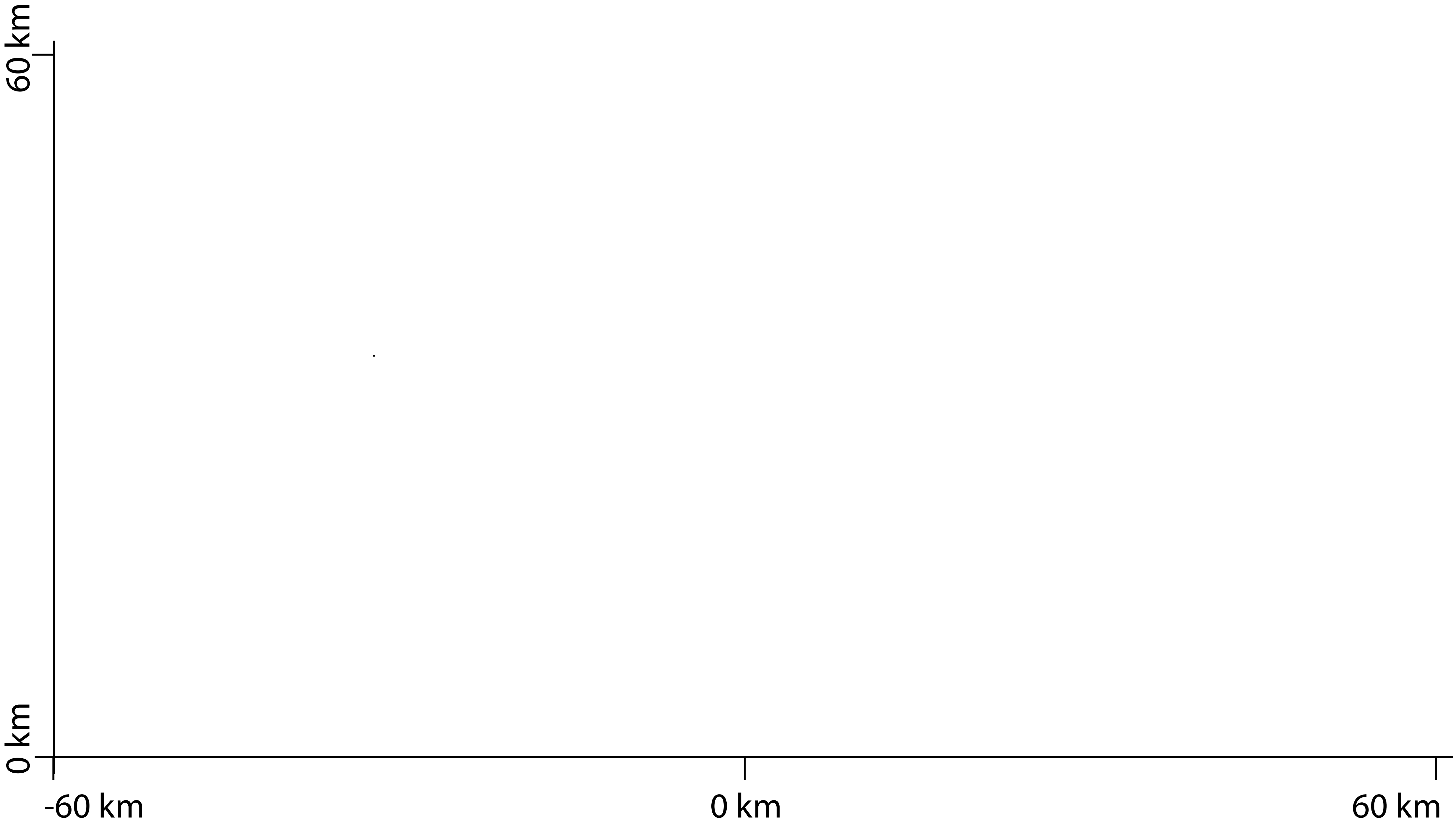


Bannon, $\mathrm{divtol} = 10^{-6}$

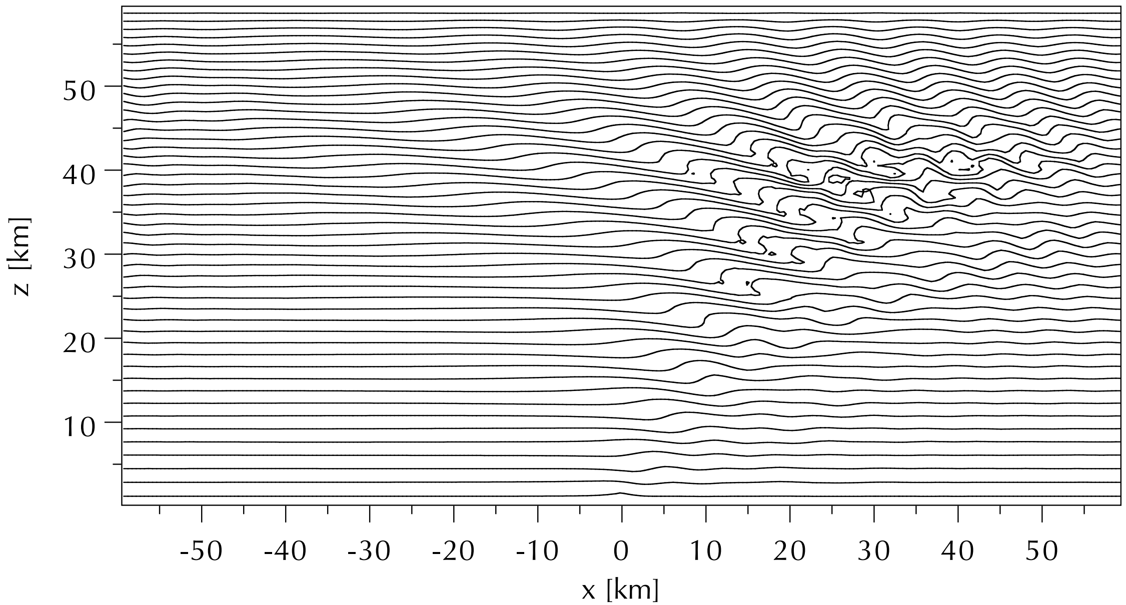
Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))

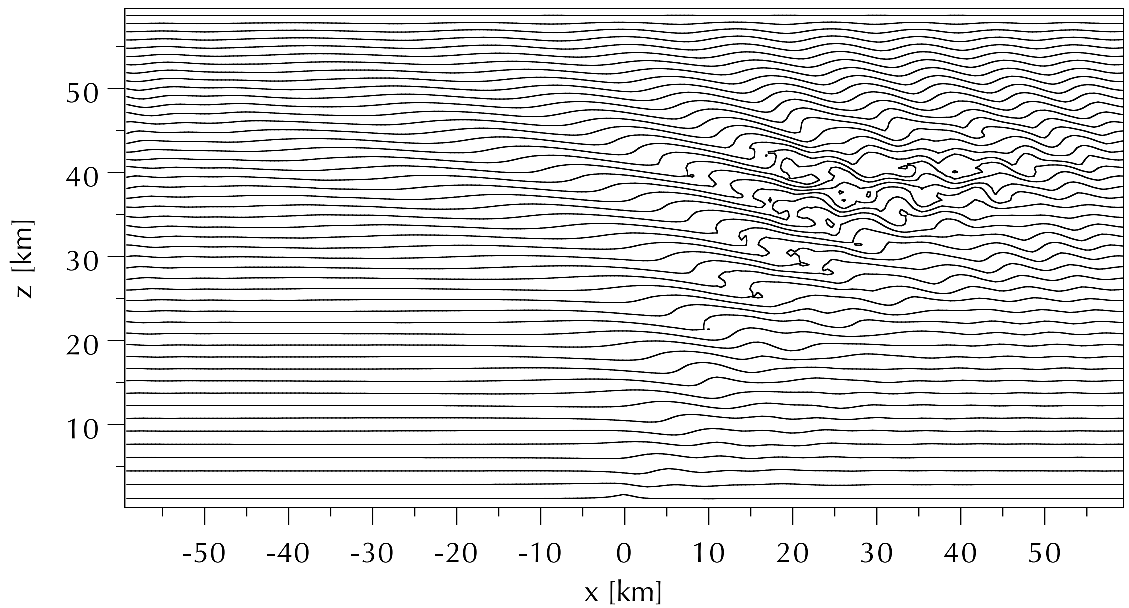


Durran's model

3 hours

sharpened van Leer's limiter

$\Delta t \, \nabla \cdot (P\boldsymbol{v}) < 10^{-3}$



Lipps-Hemler/Bannon's model

3 hours

sharpened van Leer's limiter

$\Delta t \, \nabla \cdot (P\boldsymbol{v}) < 10^{-3}$

Advection

Structure

Numerics

Asymptotics

Conclusions

Compressible flow equations:

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) \; = \; 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\mathbf{M}^2} \nabla p \; = \; -\frac{1}{\mathbf{Fr}^2} \rho \boldsymbol{k}$$

$$(\rho \theta)_t + \nabla \cdot (\rho \theta \boldsymbol{v}) \; = \; 0$$

$$\rho \theta = p^{\frac{1}{\gamma}} \,, \qquad \gamma = \frac{c_p}{c_v} \,, \qquad \mathbf{Fr} = \frac{u_{\text{ref}}}{\sqrt{g h_{\text{sc}}}} \sim \frac{u_{\text{ref}}}{\sqrt{p_{\text{ref}} / \rho_{\text{ref}}}} = \mathbf{M}$$

Asymptotics

Compressible flow equations:

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) \; = \; 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\mathbf{M}^2} \nabla p \; = \; -\frac{1}{\mathbf{M}^2} \rho g \boldsymbol{k}$$

$$\theta_t + \boldsymbol{v} \cdot \nabla \theta \; = \; 0$$

Order $O(1)$ accelerations require

$$(p, \rho, \theta) \; = \; (\bar{p}, \overline{\rho}, \overline{\theta})(z) + \mathbf{M}^2 (p', \rho', \theta')$$

$$\rho' \; = \; \overline{\rho}(z) \left(\frac{p'}{\gamma \bar{p}} - \frac{\theta'}{\overline{\theta}} \right)$$

Asymptotics

Compressible flow equations:

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) \;=\; 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \nabla p' \;=\; -\overline{\rho}(z) g \boldsymbol{k} \left(\frac{p'}{\gamma \bar{p}} - \frac{\theta'}{\overline{\theta}} \right)$$

$$\theta'_t + \boldsymbol{v} \cdot \nabla \theta' + \frac{\boldsymbol{S}}{\mathbf{M}^2} w \;=\; 0$$

$w \neq 0$ requires*

$$\boldsymbol{S} = \frac{h_{\mathrm{sc}}}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = O(\mathbf{M}^2)$$

* Ogura & Phillips (1962)

Asymptotics

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\mathrm{ref}}}{h_{\mathrm{sc}}}$	1
sound	:	$\dfrac{\sqrt{g h_{\mathrm{sc}}}}{h_{\mathrm{sc}}}$	$\dfrac{\sqrt{g h_{\mathrm{sc}}}}{u_{\mathrm{ref}}} = \dfrac{1}{\mathbf{M}}$
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\mathrm{sc}}}}{u_{\mathrm{ref}}} \sqrt{\dfrac{h_{\mathrm{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\mathbf{M}} \sqrt{\dfrac{h_{\mathrm{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$

Asymptotics

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
sound	:	$\dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\mathbf{M}}$
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\mathbf{M}} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$

For single-scale models with advection & internal waves:*

$$\frac{h_{\text{sc}}}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = O(\mathbf{M}^2) \qquad \text{or} \qquad \Delta\overline{\theta} \sim 0.3\ \text{K}$$

* Ogura & Phillips (1962)

Asymptotics & Thermodynamics

Isothermal flows $\quad (T \equiv T_{\rm ref}\,,\ h_{\rm sc} = RT_{\rm ref}/g)$

$$\frac{\overline{p}}{p_{\rm ref}} = \frac{\overline{\rho}}{\rho_{\rm ref}} = \exp\left(-\frac{z}{h_{\rm sc}}\right)\,, \qquad \frac{\overline{\theta}}{T_{\rm ref}} = \exp\left(\frac{R}{c_p}\,\frac{z}{h_{\rm sc}}\right)$$

Homentropic–to–isothermal range is accessible in an anelastic model, if

$$\Delta\overline{\theta} = O(\mathbf{M}^2)\,, \qquad \text{i.e., if*} \qquad \frac{R}{c_p} = \frac{\gamma - 1}{\gamma} = O(\mathbf{M}^2)$$

But:

- the leading-order model then is “Ogura & Phillips (1962)”
- $\overline{\theta}$–variations (e.g., in $g(\theta - \overline{\theta})/\overline{\theta}$) $\;\Rightarrow\;$ **higher-order corrections**

* Bannon (1996); Klein, Majda (2006)

Ogura & Phillips (1962):

$$
\begin{aligned}
\nabla \cdot (\overline{\rho} \boldsymbol{v}) &= 0 \\
(\overline{\rho} \boldsymbol{v})_t + \nabla \cdot (\overline{\rho} \boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho} \nabla \pi &= \overline{\rho} \theta' g \boldsymbol{k} \\
(\overline{\rho} \theta')_t + \nabla \cdot (\overline{\rho} \theta' \boldsymbol{v}) &= 0
\end{aligned}
$$

Outlook

- **Asymptotics:**
  - multiple vertical scales
  - moist processes
- **Numerics:**
  - weakly compressible motions
  - large aspect ratios
  - multiple scales